M, a point (x, y), distinct from (x_1, y_1), such that 1) $|x - x_1| < \epsilon$, 2) $x - x_1 = 0$ and $|y - y_1| < \epsilon$, in case y_1 is distinct from 0 and from 1, 3) $x \leqq x_1$ if $y_1 = 0$, 4) $x \geqq x_1$ if $y_1 = 1$.

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A PROBLEM IN THE KINEMATICS OF A RIGID BODY.

BY PROFESSOR PETER FIELD.

THE problem of finding the acceleration of any point in a rigid body when the accelerations of three points are given, and incidentally of finding what is by this means determined regarding the velocities, has received but little attention. A theorem due to Burmeister solves the problem of finding the acceleration of any point in the plane of the three points whose accelerations are given. The theorem states: "If at four coplanar points P_1, P_2, P_3, P_4 the accelerations be drawn, their extremities lie in a plane and form a quadrilateral which is affine with the quadrilateral formed by the four points."

R. Mehmke* and J. Petersen† have considered the general case, but their results do not agree, owing to an oversight in Petersen's treatment. While their work is independent, the proof in both cases depends directly on the fact that when the distance between two points is constant the projections of their velocities on their joining line are equal and the projections of their accelerations on this line differ by $\omega^2 l$, l being the distance between the two points and ω the angular velocity of the line. The purpose of this paper is to show that the problem can be solved very simply by using the expressions for the accelerations which are ordinarily given in text books on mechanics, and by this method the kinematical meaning of the solution is also evident.

Let there be given the accelerations at three points. It is proposed to find what can be determined regarding the kinematical state of the body at the given instant. As the acceleration at any point in the plane of the three points can be

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* Festschrift zur Feier des 50jährigen Bestehens der technischen Hochschule Darmstadt, page 77.

† Kinematik, page 46.

found by Burmeister's theorem, it is no restriction to assume that the points (x_0, y_0, z_0), (x_1, y_1, z_1), (x_2, y_2, z_2) whose accelerations are given have at the given instant the coordinates $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$.

The general formulas for the components $(\ddot{x}, \ddot{y}, \ddot{z})$ along the fixed axes of the acceleration of any point (x, y, z) of the moving body may be written*

$$\ddot{x} = \ddot{x}_0 + \omega_x(\omega_x x + \omega_y y + \omega_z z) - \omega^2 x + \dot{\omega}_y z - \dot{\omega}_z y,$$

$$\ddot{y} = \ddot{y}_0 + \omega_y(\omega_x x + \omega_y y + \omega_z z) - \omega^2 y + \dot{\omega}_z x - \dot{\omega}_x z,$$

$$\ddot{z} = \ddot{z}_0 + \omega_z(\omega_x x + \omega_y y + \omega_z z) - \omega^2 z + \dot{\omega}_x y - \dot{\omega}_y x.$$

In these equations ω is the angular velocity of the body, $\dot{\omega}$ the angular acceleration, and $(\ddot{x}_0, \ddot{y}_0, \ddot{z}_0)$ is the acceleration of that point in the body which at the given moment coincides with the origin of the axes of reference. These formulas applied to the points (x_1, y_1, z_1) and (x_2, y_2, z_2) give

$$\ddot{x}_1 = \ddot{x}_0 - (\omega_y^2 + \omega_z^2), \quad \ddot{y}_1 = \ddot{y}_0 + \omega_x\omega_y + \dot{\omega}_z, \quad \ddot{z}_1 = \ddot{z}_0 + \omega_x\omega_z - \dot{\omega}_y,$$

$$\ddot{x}_2 = \ddot{x}_0 + \omega_x\omega_y - \dot{\omega}_z, \quad \ddot{y}_2 = \ddot{y}_0 - (\omega_x^2 + \omega_z^2), \quad \ddot{z}_2 = \ddot{z}_0 + \omega_z\omega_y + \dot{\omega}_x.$$

These equations determine ω and $\dot{\omega}$ when the accelerations of the three points are given.

It is more convenient to say that one of the possible solutions gives p, q, r as the components of ω and l, m, n as the components of $\dot{\omega}$. Any other solution must satisfy the equations

$$q^2 + r^2 = \omega_y^2 + \omega_z^2, \quad pq + n = \omega_x\omega_y + \dot{\omega}_z, \quad pr - m = \omega_x\omega_z - \dot{\omega}_y,$$

$$pq - n = \omega_x\omega_y - \dot{\omega}_z, \quad p^2 + r^2 = \omega_x^2 + \omega_z^2, \quad qr + l = \omega_z\omega_y + \dot{\omega}_x.$$

It follows that the components of ω and $\dot{\omega}$ may be any one of the following:

I. $(p, q, r), \ (l, m, n),$

II. $(-p, -q, -r), \ (l, m, n),$

III. $(p, q, -r), \ (l + 2qr, m - 2pr, n),$

IV. $(-p, -q, r), \ (l + 2qr, m - 2pr, n).$

This shows that the absolute value of ω is determined, but

* See for instance Ziwet and Field, Introduction to Analytical Mechanics, p. 107.

the direction of the axis of spin is not. The two axes lie in a plane perpendicular to the xy plane and they make equal angles with the xy plane. The two values of the angular acceleration have the same projection along the z axis but their projections on the xy plane differ by a vector which is perpendicular to the projection of ω on this plane and equal to $2r$ times this projection. [It might be more convenient to view the two values of $\dot{\omega}$ as having the components $l + qr \pm qr,\quad m - pr \pm pr,\ n$.]

For I or II the components of the acceleration of any point (x, y, z) are

$$
\begin{aligned}
\ddot{x} &= \ddot{x}_0 - (q^2 + r^2)x + (pq - n)y + (pr + m)z, \\
\ddot{y} &= \ddot{y}_0 + (pq + n)x - (p^2 + r^2)y + (qr - l)z, \\
\ddot{z} &= \ddot{z}_0 + (pr - m)x + (qr + l)y - (p^2 + q^2)z;
\end{aligned}
\tag{1}
$$

for III or IV they are

$$
\begin{aligned}
\ddot{x} &= \ddot{x}_0 - (q^2 + r^2)x + (pq - n)y + (m - 3pr)z, \\
\ddot{y} &= \ddot{y}_0 + (pq + n)x - (p^2 + r^2)y - (l + 3qr)z, \\
\ddot{z} &= \ddot{z}_0 + (pr - m)x + (l + qr)y - (p^2 + q^2)z.
\end{aligned}
\tag{2}
$$

It is no restriction to take the axis of rotation in the yz plane, i. e., $p = 0$. In place of (1) and (2) we then have (1′) and (2′)

$$
\begin{aligned}
\ddot{x} &= \ddot{x}_0 - (q^2 + r^2)x - ny + mz, \\
\ddot{y} &= \ddot{y}_0 + nx - r^2y + (qr - l)z, \\
\ddot{z} &= \ddot{z}_0 - mx + (qr + l)y - q^2z,
\end{aligned}
\tag{1′}
$$

and

$$
\begin{aligned}
\ddot{x} &= \ddot{x}_0 - (q^2 + r^2)x - ny + mz, \\
\ddot{y} &= \ddot{y}_0 + nx - r^2y - (l + 3qr)z, \\
\ddot{z} &= \ddot{z}_0 - mx + (l + qr)y - q^2z.
\end{aligned}
\tag{2′}
$$

These equations show that there are two possible values for the acceleration of any point not in the xy plane. These values become equal if either q or r is equal to zero; i. e., if the axis of spin is either parallel or perpendicular to the plane of the three points. If neither q nor r is equal to zero, the center of acceleration is different for the two cases unless it should happen to lie in the xy plane.

Summary.—When the accelerations of three points in a rigid body are given, the acceleration of any point in the plane of the given points is determined uniquely. The acceleration of a point not in the plane of the given points is in general two valued. Moreover, there are in general four sets of values of ω and $\dot{\omega}$ which give the same values for the accelerations of the points in a given plane. For a given value of ω there can be determined the value of the spin for the line joining a given pair of points and hence the relative velocity of the two points can be found.

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September, 1915.

JULES HENRI POINCARÉ.

Enquête de "l'Enseignement Mathématique" sur la méthode de travail des mathématiciens. Publié par H. FEHR avec la collaboration de T. FLOURNOY et E. CLAPARÈDE. Deuxième édition conforme à la première suivie d'une *Note sur l'invention mathématique* par H. POINCARÉ. Paris, Gauthier-Villars, et Genève, George, 1912. 8vo. 8+137 pages. Price 5 francs.

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Savants du Jour: Henri Poincaré, Biographie, Bibliographie analytique des écrits. Seconde édition entièrement refondue. Par E. LEBON. Paris, Gauthier-Villars, 1912. Royal 8vo. 112 pages. Price 7 francs.

IT was in the latter part of 1900 that M. E. Maillet wrote as follows:* "Messieurs les Rédacteurs, Il y aurait, ce me semble, une tentative à faire, pour laquelle l'*Enseignement Mathématique* est à mon avis tout à fait désigné, et dont le succès pourrait rendre de bien grands services aux jeunes mathématiciens. Elle consisterait à ouvrir une sorte d'enquête auprès de savants connus; il s'agirait d'obtenir de chacun d'eux quelques renseignements personnels sur sa méthode de travail et de recherche, ses habitudes, l'hygiène générale qu'il juge la plus propre à faciliter son travail intellectuel, la manière de conduire le plus efficacement ses lectures et d'en tirer le meilleur parti, etc., etc. Je me borne ici à indiquer les grandes

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* *L'Enseignement Mathématique*, 1901, tome 3, p. 58.