of the determinant $\Delta(x, \xi)$ is an isolated one. This is the second theorem of Bolza mentioned in the introduction above.

The formula (5) is identical with a formula of von Escherich* when the values of c from equations (4) are substituted and η replaced by z. The proof here is, however, of an entirely different character and by far more simple than his. Bolza uses the formula of von Escherich for the purpose of transforming the second variation, and with the help of this transformation deduces the theorem last given. This process seems very much less direct than the argument given above.

UNIVERSITY OF CHICAGO.

CONCERNING A NON-METRICAL PSEUDO-ARCHIMEDEAN AXIOM.

BY DR. ROBERT L. MOORE.

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§ 1. *Introduction.*

LET H_1 denote Hilbert's *plane* Axioms of Groups I and II† or Veblen's Axioms I–VIII.‡ Let H_2 denote H_1 together with Desargues' theorem§ (considered as an axiom) and Hilbert's III (axiom of parallels). It is well known that if a two-dimensional space S satisfies H_2 together with Hilbert's congruence axioms of Group IV and the archimedean axiom that of any two non-congruent segments some multiple of the smaller is larger than the greater, then S is either an ordinary euclidean space of two dimensions or an everywhere dense subset of such a space.

Consider the following non-metrical pseudo-archimedean axiom:

AXIOM A. *If* (1) *the points of a line l (Fig. 1) are divided into two sets S_1 and S_2 such that no point of either of these sets is between two points of the other one and such that no point P is*

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* Loc. cit., p. 1283, formula (9); Bolza, loc. cit., p. 630, formula (68).

† D. Hilbert, Foundations of Geometry, translated by E. J. Townsend, Open Court Publishing Co., Chicago, 1902.

‡ O. Veblen, "A system of axioms for geometry," *Transactions Amer. Math. Society*, vol. 5 (1904), pp. 343–384.

§ Cf. Hilbert, loc. cit., p. 71.

between every point of S_1 *distinct from* P *and every point of* S_2 *distinct from* P, (2) A *and* B *are distinct points on the same side of* l, (3) t *is a triangle whose interior contains a point of* S_1 *and a point of* S_2;

then there exists within t, *and on the far side of* l *from* A *and* B, *a point* C *such that the interior of the triangle* ABC *contains a point of* S_1 *and a point of* S_2.

In the present paper it will be shown that every two-

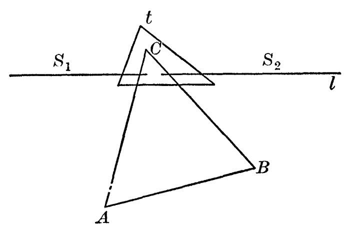

FIG. 1.

dimensional space that satisfies H_1 and Axiom A is equivalent, from the standpoint of analysis situs, either to an ordinary euclidean space of two dimensions or to an everywhere dense subset of such a space. For a precise formulation of this proposition and for a theorem concerning the part that Axiom A plays in connection with the system H_2, the reader is referred to § 3.*

In view of these results it is clear that the non-metrical Axiom A plays a rôle which is, in certain respects, analogous to that played by the above mentioned *metrical* archimedean axiom.

§ 2. *Deductions from* H_1 *and Axiom* A.

THEOREM 1. *If* P *is a point on a line* l, *then there exist on* l *two countably infinite sequences of points* $A_1, A_2, A_3, \cdots$ *and* $B_1, B_2, B_3, \cdots$ *such that* (1) *for every* m *and* n, P *is between* A_m *and* B_n, (2) A_{n+1} *and* B_{n+1} *are between* A_n *and* B_n, (3) *if* $P' \neq P$ *there exists an* n *such that the segment* A_nB_n *does not contain* P'.

Proof. There exist points O, M_1, N, A and C such that O is on l but distinct from P, M_1 is not on l, P is between M_1 and N, A is within the angle NPO and C is between A and

* With regard to another form of non-metrical pseudo-archimedean axiom, see references to Vahlen in §§ 2 and 3 below.

P. The segment AM_1 contains a point A_1 in common with the ray PO. The segments CM_1 and PA_1 have a point A_2 in common. There exists a point M_2 on the segment PM_1 and in the order AA_2M_2. The segments CM_2 and PA_2 have in common a point A_3. Continue this process. In general, for every n the following orders hold: AA_nM_n, $PA_{n+1}A_n$, $CA_{n+1}M_n$. Suppose there exists, on the ray PO, a point X such that no point of the sequence $A_1, A_2, A_3, \cdots$ is between P and X. Let S_1 denote the set of all such points $[X]$ together with all points $[Y]$ such that, for some X, Y is on the ray XP. Let S_2 denote the set of all other points of the line l. With the assistance of Axiom A it can be shown that there exists, within the angle OPM_1, a point Z such that the interior of the triangle ACZ contains at least one point K belonging to S_1 and at least one point A_m of the sequence $A_1, A_2, A_3, \cdots$. It is clear that A_{m+1} is between P and K. Thus the supposition that there exists a point X as described above leads to a contradiction. It follows that, for every point P' on the ray PO, the segment PP' contains a point of the sequence $A_1, A_2, A_3, \cdots$.* Similarly, there exists on the ray PO' (where O' is a point in the order OPO') a sequence of points $B_1, B_2, B_3, \cdots$ such that, for every n, B_{n+1} is between B_n and P and such that if P' is a point on the ray PO' then the segment PP' contains a point of the sequence $B_1, B_2, B_3, \cdots$. It is clear that the sequences $A_1, A_2, A_3, \cdots$ and $B_1, B_2, B_3, \cdots$ satisfy conditions (1), (2) and (3).

THEOREM 2. *If the points of the line l' are divided into two sets S_1 and S_2 such that no point of either of these sets is between two points of the other one, then there exist two sequences of points $A_1', A_2', A_3', \cdots$ and $B_1', B_2', B_3', \cdots$ such that* (*a*) *every A_n' belongs to S_1 and every B_n' belongs to S_2,* (*b*) *for every n the points A_{n+1}' and B_{n+1}' are between A_n' and B_n',* (*c*) *if C' and D' are distinct points on l', there exists n such that the segment $A_n'B_n'$ does not contain both C' and D'.*

Proof. There exist two points A and B lying on the same side of l' (Fig. 2). Between A and B there is a point P. Let l denote the line AB. By Theorem 1 there exist, on the rays

* In connection with two or three theorems in his paper "Curves in non-metrical analysis situs with an application in the calculus of variations," Lennes makes use of an axiom (which I will call Axiom B) to the effect that P is a limit point of the sequence $A_1, A_2, A_3, \cdots$. Cf. *Amer. Jour. of Mathematics*, vol. 33 (1911), p. 305. Cf. also K. T. Vahlen, Abstrakte Geometrie, Leipzig, 1905, p. 156.

PA and PB respectively, sequences $A_1, A_2, A_3, \cdots$ and $B_1, B_2, B_3, \cdots$ satisfying conditions (1), (2) and (3). By Axiom A there exists, on the remote side of l' from A, a point O_1 such that the interior of the triangle $A_1O_1B_1$ contains a point A_1' of S_1 and a point B_1' of S_2. Within the triangle $A_1'O_1B_1'$ there is a point O_2 such that the interior of the triangle $A_2O_2B_2$

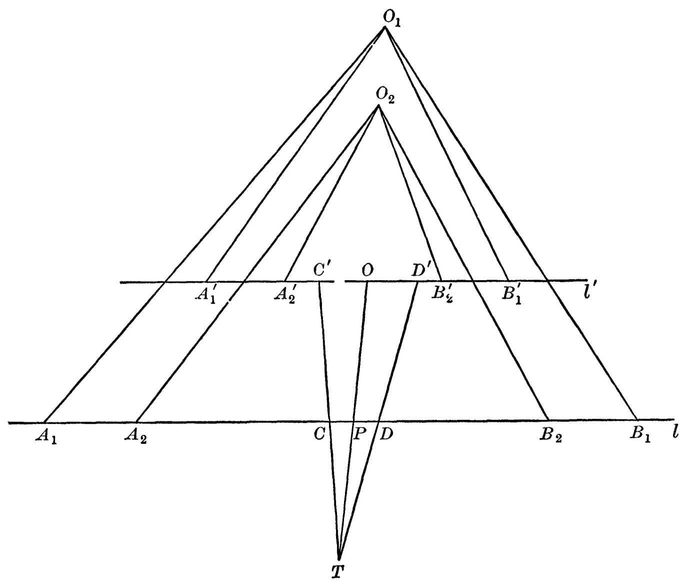

FIG. 2.

contains a point A_2' of S_1 and a point B_2' of S_2 in the order $A_1'A_2'B_2'B_1'$. Continue this process. In general, the point O_{n+1} is within the triangle $A_n'O_nB_n'$, the points A_n' and B_n' belong to S_1 and S_2 respectively and are both within the triangle $A_nO_nB_n$, and the points A_n', A_{n+1}', B_n', B_{n+1}' are in the order $A_n'A_{n+1}'B_{n+1}'B_n'$. It is clear that the sequences $A_1', A_2', A_3', \cdots$ and $B_1', B_2', B_3', \cdots$ satisfy conditions (a) and (b). That they satisfy condition (c) may be proved as follows.

Suppose that they do not satisfy condition (c). Then there exist, on l', two distinct points C' and D' such that for every n the segment $A_n'B_n'$ contains both C' and D' and such that D' is between C' and every B_n'. But between C' and D'

there is a point O. There exists a point T in the order OPT. There exist on the line l points C and D in the orders TCC' and TDD'. There exists m such that A_m and B_m are both between C and D. If neither A_m' nor B_m' is between C' and D' then the lines A_mA_m' and B_mB_m' intersect in a point within or on the triangle TCD. But by hypothesis they intersect on the remote side of l' from A and B. Thus the supposition that condition (c) is not satisfied here leads to a contradiction.

DEFINITION 1. Two segments AB and CD are said to be *separated* if no point or end point of AB is either a point or an end point of CD. Two triangles are said to be *separated* if no point of either of them is on or within the other one.

DEFINITION 2. Suppose $t_1, t_2, t_3, \cdots$ is a countable sequence of triangles such that (1) for every n, t_{n+1} is within t_n, (2) if each of the segments AB and CD intersects every t_n, then AB and CD are not separated. Then the set of all triangles $[t]$ such that the interior of t contains some triangle of the sequence $t_1, t_2, t_3, \cdots$ is called an *ideal point*. The sequence $t_1, t_2, t_3, \cdots$ is said to be a *fundamental sequence* for this ideal point. If α is an ideal point, t_α denotes one of the triangles of which α is composed.

DEFINITION 3. If the line l intersects every triangle of the ideal point α then l is said to *contain* α, and α is said to *lie on* l. If some triangle of α lies on a given side of l then α is said to lie on that side of l.

THEOREM 3. *If α and β are distinct ideal points then there exist two triangles which belong to α and β respectively and are separated from each other.*

THEOREM 4. *If t_α' and t_α'' are triangles belonging to the ideal point α then there exists a triangle t_α''' belonging to α and lying within both t_α' and t_α''.*

THEOREM 5. *If A is a real point, the set of all triangles whose interiors contain A is an ideal point.**

THEOREM 6. *If the points of the line l' are divided into two sets S_1 and S_2 such that no point of either of these sets is between two points of the other one then there exists an ideal point α such that every triangle of α contains a point of S_1 and a point of S_2.*

Proof. There exist, in S_1 and S_2 respectively, sequences

* The ideal point which is determined in this way by the real point A will be denoted by the symbol A^*.

A_1', A_2', A_3', $\cdots$ and B_1', B_2', B_3', $\cdots$ satisfying conditions (*a*), (*b*) and (*c*) of Theorem 2. There exist points P and $\bar{P}$ lying on opposite sides of l'. It follows with the help of Theorem 1 that there exist on the segments PA_1' and PB_1' respectively sequences of points A_1, A_2, A_3, $\cdots$ and B_1, B_2, B_3, $\cdots$ in the orders $PA_1A_2\cdots A_1'$ and $PB_1B_2\cdots B_1'$ and such that if X is on the ray $A_1'P$ and Y is on the ray $B_1'P$ then there exists m such that A_m is between X and A_1' while B_m is between Y and B_1'. Similarly there exist, on the segments $\bar{P}A_1'$ and $\bar{P}B_1'$ respectively, sequences $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3$, $\cdots$ and $\bar{B}_1$, $\bar{B}_2$, $\bar{B}_3$, $\cdots$ in the orders $\bar{P}\bar{A}_1\bar{A}_2\cdots A_1'$ and $\bar{P}\bar{B}_1\bar{B}_2\cdots B_1'$ and such that if X is on the ray $A_1'\bar{P}$ and Y is on the ray $B_1'\bar{P}$ then there exists m such that $\bar{A}_m$ is between X and A_1' while $\bar{B}_m$ is between Y and B_1'. For each n the segments $A_n'B_n$ and $B_n'A_n$ have a point O_n in common. There exist points C_n and D_n in the orders $O_nA_{n+1}'C_n$, $A_n'C_n\bar{B}_n$, $O_nB_{n+1}'D_n$, $B_n'D_n\bar{A}_n$. Let t_n denote the triangle $O_nC_nD_n$. Let α denote the set of all triangles $[t]$ such that, for some n, t_n is within t. It can be proved that α is an ideal point. Every triangle of α contains a point of S_1 and a point of S_2.

THEOREM 7. *If the real points A and B and the ideal point α all lie on the same side of the line l and α is not on the line AB then there exists a point C, on the same side of l as A, such that the interior of the triangle ABC contains some triangle of α.*

Proof. Suppose there exists no such point C. If t is any triangle of α which has no point in common with the line AB, there exist lines a_t and b_t containing A and B respectively such that (1) the interior of t is entirely on the A-side of b_t and entirely on the B-side of a_t, (2) the perimeter of t contains a point of a_t and also a point of b_t. Let $\bar{t}$ denote a definite triangle of α whose perimeter has no point in common with either l or AB. Let t_1, t_2, t_3, $\cdots$ be a fundamental sequence of triangles of α such that t_1 lies within $\bar{t}$. For every triangle t that lies within t_1, a_t intersects the perimeter of t_1 in two points A_t and A_t' in the order $AA_t'A_t$ while b_t intersects the perimeter of t_1 in points B_t and B_t' in the order $BB_t'B_t$. Let $A_tY_tB_t$ denote that arc* of the perimeter of t_1 whose end points are A_t and B_t and which contains neither A_t' nor B_t'. Then clearly, for every n, $A_{t_n}Y_{t_n}B_{t_n}$ contains $A_{t_{n+1}}Y_{t_{n+1}}B_{t_{n+1}}$. There are two cases to be considered.

* This arc is either a segment or a broken line.

Case I. Suppose there exist two points $\bar{A}$ and $\bar{B}$ such that on $A_{t_2}Y_{t_2}B_{t_2}$ the order $A_{t_2}A_{t_n}\bar{A}\bar{B}B_{t_n}B_{t_2}$ holds true for every n. Then each of the two segments $A\bar{A}$ and $B\bar{B}$ intersects every t_n. Thus a contradiction is reached.

Case II. Suppose there do not exist two such points $\bar{A}$ and $\bar{B}$. Then it may be easily proved with the help of Axiom A that there exists, within the triangle $\bar{t}$, a point C such that AC and BC intersect t_1 in points A' and B' respectively such that, for some n, the order $A_{t_2}A'A_{t_n}B_{t_n}B'B_{t_2}$ holds true and thus t_n lies within the triangle ABC. Thus again a contradiction is obtained.

THEOREM 8. *If α and β are two distinct ideal points which do not both lie on the line l then there do not exist on l two distinct points A and B such that for every t_α, t_β and point P of the segment AB there is a line through P that intersects t_α and t_β.*

Proof. Suppose there exists such a segment AB. Suppose α is not on l (Fig. 3). There exist two separated triangles $\bar{t}_\alpha$

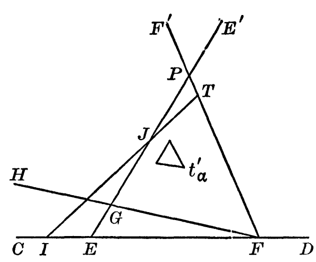

FIG. 3.

and $\bar{t}_\beta$ belonging to α and β respectively. There exists a line $\bar{l}$ distinct from l such that $\bar{t}_\alpha$ and $\bar{t}_\beta$ lie on different sides of $\bar{l}$. There exist, on the segment AB, points C and D such that $\bar{l}$ contains neither C nor D nor any point between C and D. There exist E and F in the order $CEFD$. Suppose* $\bar{t}_\alpha$ is on the same side of $\bar{l}$ as E. By Theorem 7 there exists a point P on the E-side of $\bar{l}$ and a triangle t_α' belonging to α such that t_α' lies entirely within the triangle EPF. Clearly $\bar{t}_\beta$ is entirely without the triangle EPF. There exist points

*Every other case may be either easily disposed of or reduced to a case which is, apart from notation, the same as this one.

E' and F' in the orders EPE' and FPF'. If every triangle of β contained a point within the angle EPF' and also a point within the angle FPE' then each of the lines EP and FP would intersect every triangle of β. But not every triangle of β contains P. Thus β would not be an ideal point according to Definition 2. It follows that one of the angles EPF' and FPE' is such that some triangle of β contains no point within or on that angle. Suppose this is true of the angle FPE'. Then, by Theorem 4, there exists a triangle belonging to β and containing no point within or on the triangle EPF or within or on the angle FPE'. There exists, between E and P, a po nt G such that no point of t_α' is within the triangle FEG. But for every t_β there is a line passing through F and intersecting t_α' and t_β. It follows that every t_β contains a point within the angle HGP where H is a point in the order FGH. Hence there exists a triangle t_β' belonging to β and lying entirely within the angle CEP. There exist points I, J and T in the orders CIE, EJP, FTP and IJT and such that t_α' is within the triangle ITF while t_β' is within the angle IJP. But I is collinear with a point of t_β' and a point of t_α'. It follows that t_β' contains a point within the triangle IJE. But t_β' is entirely within the angle IJP. Thus the supposition that Theorem 8 is false leads to a contradiction.

DEFINITION 4. If α and β are distinct ideal points, the *ideal line* $\alpha\beta$ is the set of all ideal points $[\gamma]$ such that for every t_α, t_β, t_γ there exist three collinear real points A, B, C such that A is within t_α, B is within t_β and C is within t_γ. If α, β and γ are distinct ideal points such that for every t_α, t_β, t_γ there exist real points A, B and C within t_α, t_β and t_γ respectively and in the order ABC then α β and γ are said to be *in the order* $\alpha\beta\gamma$.

THEOREM 9. *If A and B are two distinct real points and α and β are two distinct ideal points such that neither A^* nor B^* is on the ideal line $\alpha\beta$, and if, furthermore, for every two triang es t_α and t_β, belonging to α and β respectively, there exists, between A and B, a point which is collinear with some point of t_α and some point of t_β then there exist t_α and t_β such that every point of l which is collinear with a point of t_α and a point of t_β lies be- ween A and B and there exists, on the ideal line $\alpha\beta$, an ideal point γ in the order $A^*\gamma B^*$.*

THEOREM 10. *If α and β are distinct ideal points, there exists an ideal point γ in the order $\alpha\beta\gamma$.*

THEOREM 11. *If α, β and γ are three ideal points in the order $\alpha\beta\gamma$ and $\bar{t}_\alpha$, $\bar{t}_\beta$, $\bar{t}_\gamma$ are mutually separated triangles belonging to α, β and γ respectively, then there exists t_β such that* (1) *every point of t_β is collinear with some point of $\bar{t}_\alpha$ and some point of $\bar{t}_\gamma$,* (2) *if P_α, P_β, P_γ are collinear points belonging to $\bar{t}_\alpha$, t_β and $\bar{t}_\gamma$ respectively then P_α, P_β, P_γ are in the order $P_\alpha P_\beta P_\gamma$.*

THEOREM 12. *If the ideal points α, β, γ are in the order $\alpha\beta\gamma$ then they are not in the order $\beta\gamma\alpha$.*

THEOREM 13. *If the ideal point β lies on the ideal line $\alpha\gamma$ then for every t_β there exist t_α and t_γ such that every line that intersects t_α and t_γ intersects also t_β.*

Proof. There exists, within t_β, a triangle t_β' belonging to β. For every t_α and t_γ there is a line intersecting t_α, t_γ and t_β'. It follows with the aid of Theorem 9 that, for some lettering A, B, C of the vertices of t_β', it is true that, for every point X in the order ABX there exist triangles $t_{\alpha X}$ and $t_{\gamma X}$ belonging to α and γ respectively such that every line that intersects $t_{\alpha X}$ and $t_{\gamma X}$ intersects also the segment AX. But X may be chosen within the triangle t_β. Thus there exist t_α and t_γ such that every line that intersects both t_α and t_γ intersects also t_β.

THEOREM 14. If γ_1, γ_2, γ_3 *are three distinct ideal points lying on the ideal line $\alpha\beta$ then γ_1 is on the ideal line $\gamma_2\gamma_3$.*

Proof. Suppose t_{γ_1}, t_{γ_2} and t_{γ_3} are triangles belonging to γ_1, γ_2 and γ_3 respectively. By Theorem 13 there exist triangles $t_\alpha^{(1)}$, $t_\alpha^{(2)}$, $t_\alpha^{(3)}$, belonging to α, and triangles $t_\beta^{(1)}$, $t_\beta^{(2)}$, $t_\beta^{(3)}$, belonging to β, such that if $i = 1$, 2 or 3 then every line that intersects $t_\alpha^{(i)}$ and $t_\beta^{(i)}$ intersects also t_{γ_i}. But by Theorem 4 there exist $\bar{t}_\alpha$ and $\bar{t}_\beta$ belonging to α and β respectively such that $\bar{t}_\alpha$ is within every $t_\alpha^{(i)}$ and $\bar{t}_\beta$ is within every $t_\beta^{(i)}$ ($i = 1, 2, 3$). Every line that intersects $\bar{t}_\alpha$ and $\bar{t}_\beta$ must intersect t_{γ_1}, t_{γ_2} and t_{γ_3}. It follows that γ_1 is on $\gamma_2\gamma_3$.

THEOREM 15. *If α, β, γ and δ are four distinct ideal points no three of which are collinear* and for every t_α, t_β, t_γ, t_δ there exists a point which is collinear with a point of t_α and a point of t_β and which at the same time is between a point of t_γ and a point of t_δ, then there exists, on the ideal line $\alpha\beta$, an ideal point ϵ in the order $\gamma\epsilon\delta$.*

Proof. Since no three of the points α, β, γ and δ are

* Three or more ideal points are said to be collinear if there exists an ideal line which contains them all.

collinear there exist four mutually separated triangles $\bar{t}_\alpha$, $\bar{t}_\beta$, $\bar{t}_\gamma$, $\bar{t}_\delta$ belonging to α, β, γ, δ respectively and such that no line intersecting $\bar{t}_\alpha$ and $\bar{t}_\beta$ intersects either $\bar{t}_\gamma$ or $\bar{t}_\delta$. Let $t_{\alpha 1}$, $t_{\alpha 2}$, $t_{\alpha 3}$, $\cdots$, $t_{\beta 1}$, $t_{\beta 2}$, $t_{\beta 3}$, $\cdots$, $t_{\gamma 1}$, $t_{\gamma 2}$, $t_{\gamma 3}$, $\cdots$, $t_{\delta 1}$, $t_{\delta 2}$, $t_{\delta 3}$, $\cdots$ be fundamental sequences belonging to α, β, γ, δ respectively and such that $t_{\alpha 1}$, $t_{\beta 1}$, $t_{\gamma 1}$, $t_{\delta 1}$ lie within $\bar{t}_\alpha$, $\bar{t}_\beta$, $\bar{t}_\gamma$, $\bar{t}_\delta$ respectively. For each positive integer n there exists a convex polygon p_n, of six sides or less, such that (1) with the exception of two sides, every side of p_n is a side of $t_{\gamma n}$ or a side of $t_{\delta n}$, (2) each of the two remaining sides of p_n has for one end point a point of $t_{\gamma n}$ and for its other end point a point of $t_{\delta n}$, (3) the interior of p_n contains the interiors of $t_{\gamma n}$ and $t_{\delta n}$. By hypothesis there exists, within p_n, a point X_n which is collinear with a point of $t_{\alpha n}$ and a point of $t_{\beta n}$. It follows that there exists a quadrilateral $A_nB_nC_nD_n$ such that (1) every point on or within $A_nB_nC_nD_n$ is between a point of $t_{\gamma n}$ and a point of $t_{\delta n}$ and is also on some line that intersects $t_{\alpha n}$ and $t_{\beta n}$, (2) every point which is common to a line intersecting $t_{\alpha n}$ and $t_{\beta n}$ and a line intersecting $t_{\gamma n}$ and $t_{\delta n}$ is on or within $A_nB_nC_nD_n$. If there exists m such that for every n greater than m the interior of $A_nB_nC_nD_n$ contains a point of the diagonal A_mC_m then it follows with the help of Theorem 9 that there exists, on A_mC_m, an ideal point which is collinear with α and β and, at the same time, is between γ and δ. If there exists no such m then the sequence of triangles $A_1B_1C_1$, $A_1C_1D_1$, $A_2B_2C_2$, $A_2C_2D_2$, $\cdots$, contains, as a subsequence, an infinite sequence t_1, t_2, t_3, $\cdots$ such that, for every n, t_{n+1} is within t_n. There do not exist two separated segments each of which contains a point of every t_n. For if there did exist two such segments, s_1 and s_2, they would* contain two ideal points ϵ_1 and ϵ_2 respectively such that ϵ_1 and ϵ_2 are both on $\alpha\beta$ and both on $\gamma\delta$ and therefore, by Theorem 14, α, β, γ and δ would be collinear, which is contrary to hypothesis. Hence if ϵ denotes the set of all triangles $[t]$ such that the interior of t contains at least one triangle of the sequence t_1, t_2, t_3, $\cdots$ then ϵ is an ideal point. It is clear that ϵ is between γ and δ and collinear with α and β.

THEOREM 16. *If α, β and γ are non-collinear ideal points and δ and ϵ are two ideal points in the orders $\alpha\beta\delta$ and $\beta\epsilon\gamma$ then there exists, on the ideal line $\delta\epsilon$, an ideal point η in the order $\gamma\eta\alpha$.*

Proof. There exist mutually separated triangles $t_{\alpha 1}$, $t_{\beta 1}$,

* Cf. Theorem 9.

$t_{\gamma 1}$, $t_{\delta 1}$, $t_{\epsilon 1}$, belonging to α, β, γ, δ, ϵ respectively and such that no point of $t_{\gamma 1}$ is on a line intersecting $t_{\alpha 1}$ and $t_{\beta 1}$. If $t_{\alpha 2}$ and $t_{\delta 2}$ are triangles of α and δ respectively lying within $t_{\alpha 1}$ and $t_{\delta 1}$ respectively then, by Theorem 11, there exists, within $t_{\beta 1}$, a triangle $t_{\beta 2}$ belonging to β and such that every point of $t_{\beta 2}$ is between some point of $t_{\alpha 2}$ and some point of $t_{\delta 2}$. Let $t_{\gamma 2}$ and $t_{\epsilon 2}$ denote triangles belonging to γ and ϵ respectively. By hypothesis there exist within $t_{\beta 2}$, $t_{\epsilon 2}$ and $t_{\gamma 2}$, three points, P_β, P_ϵ and P_γ respectively, in the order $P_\beta P_\epsilon P_\gamma$. There exist points P_α and P_δ within $t_{\alpha 2}$ and $t_{\delta 2}$ respectively and in the order $P_\alpha P_\beta P_\delta$. Since P_α, P_β and P_γ are not collinear and furthermore P_β is between P_α and P_δ while P_ϵ is between P_β and P_γ, it follows that there exists a point P_η in the orders $P_\delta P_\epsilon P_\eta$ and $P_\gamma P_\eta P_\alpha$. It follows by Theorem 15 that there exists, on the ideal line $\delta\epsilon$, an ideal point η in the order $\gamma\eta\alpha$.

THEOREM 17. *If the points of an ideal line are divided into two sets S_1 and S_2 such that no point of either of these sets is between two points of the other one, then there exists an ideal point α which lies between every point of S_1 distinct from α and every point of S_2 distinct from α.*

THEOREM 18. *The set of all ideal points satisfies Veblen's Axioms I–VIII, XI (also Hilbert's plane axioms of Groups I and II together with the Dedekind cut postulate).*

§ 3. *Conclusion.*

DEFINITION. A space S consisting of a definite system of *points* and *lines* subject to definite relations of *alignment* and *order* is said to be descriptively equivalent to a subset S' of an ordinary euclidean space E if there exists between the *points* of S and the points of S' a one-to-one reciprocal correspondence preserving collinearity and order.*

DEFINITION. A two-dimensional space S consisting of a definite system of *points* and *lines* subject to definite relations of *alignment* and *order* is said to be equivalent, from the standpoint of analysis situs, to a subset S' of a two-dimensional euclidean space E if there exists, between the *points* of S

* The statement that such a correspondence preserves collinearity and order signifies that if A, B, C are three *points* of S and A', B', C' respectively are the corresponding points of S' then A, B, C are in the *order* ABC on a line in S if and only if A', B', C' are in the order $A'B'C'$ on a line in E.

and the points of S', a one-to-one reciprocal correspondence preserving limits.*

The following theorems may be easily established with the assistance of Theorem 18 of § 2 and Theorem IV of my paper "On a set of postulates which suffice to define a number-plane."†

THEOREM A. *Every two-dimensional space that satisfies Hilbert's plane axioms of Groups I and II (or Veblen's I–VIII) together with Axiom A is equivalent, from the standpoint of analysis situs, either to a two-dimensional euclidean space or to an everywhere dense subset thereof.*

THEOREM B.‡ *Every two-dimensional space that satisfies Hilbert's plane axioms of Groups I, II and III (or Veblen's I–VIII, XII) together with Desargues' theorem and Axiom A is descriptively equivalent either to a two-dimensional euclidean space or an everywhere dense subset thereof.*

COROLLARY. *Pascal's theorem§ is a consequence of Hilbert's plane axioms of Groups I, II and III together with Desargues' theorem and Axiom A.*

THE UNIVERSITY OF PENNSYLVANIA.

A TYPE OF SINGULAR POINTS FOR A TRANSFORMATION OF THREE VARIABLES.

BY DR. W. V. LOVITT.

(Read before the American Mathematical Society, December 31, 1915.)

IN the *Transactions* for October, 1915, I discussed some singularities of a point transformation in three variables

$$(1) \qquad x = \phi(u, v, w), \quad y = \psi(u, v, w), \quad z = \chi(u, v, w)$$

* The statement that such a correspondence preserves limits signifies that if A is a *point* of S, M is a *point set* of S, and A' and M' respectively are the corresponding point and point set of S' then P is a *limit point* of M if, and only if, P' is a limit point of M'. Here P is said to be a *limit point* of M if, and only if, every *triangle* of S that contains P *within* it contains *within* it at least one *point* of M distinct from P.

† *Transactions of the American Mathematical Society*, vol. 16 (1915), pp. 27–32.

‡ For a corresponding theorem regarding Axiom B (cf. footnote in § 2) see Vahlen, loc. cit., pp. 158–163.

§ Cf. Hilbert, loc. cit., p. 40.