THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The one hundred and eighty-second regular meeting of the Society was held in New York City on Saturday, February 26, 1916. The attendance at the morning and afternoon sessions included the following forty-three members:

Mr. J. W. Alexander, II, Mr. W. E. Anderson, Mr. D. R. Belcher, Dr. A. A. Bennett, Professor Joseph Bowden, Professor E. W. Brown, Professor J. G. Coffin, Professor F. N. Cole, Professor Elizabeth B. Cowley, Professor Louise D. Cummings, Dr. H. B. Curtis, Professor L. P. Eisenhart, Professor H. B. Fine, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Professor O. E. Glenn, Dr. T. H. Gronwall, Professor C. C. Grove, Professor H. E. Hawkes, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Mr. Harry Langman, Mr. G. H. Light, Mr. P. H. Linehan, Professor E. J. Miles, Mr. G. W. Mullins, Mr. George Paaswell, Dr. P. R. Rider, Mr. J. F. Ritt, Dr. Caroline E. Seely, Dr. H. M. Sheffer, Professor L. P. Siceloff, Professor Mary E. Sinclair, Professor P. F. Smith, Mr. C. E. Van Orstrand, Professor Oswald Veblen, Mr. H. E. Webb, Dr. Mary E. Wells, Professor H. S. White, Mr. J. K. Whittemore, Miss E. C. Williams.

The President of the Society, Professor E. W. Brown, occupied the chair, being relieved by Professors Fine, Fiske, and White. The Council announced the election of the following persons to membership in the Society: Mr. L. E. Armstrong, Stevens Institute of Technology; Professor Grace M. Bareis, Ohio State University; Professor G. A. Chaney, Iowa State College; Mr. J. E. Davis, Pennsylvania State College; G. H. Hardy, M.A., Trinity College, Cambridge, England; Mr. Harry Langman, Metropolitan Life Insurance Company, New York City; Mr. E. D. Meacham, University of Oklahoma; Dr. A. L. Nelson, University of Michigan; Mr. Elmer Schuyler, Bay Ridge High School, Brooklyn, N. Y. Six applications for membership in the Society were received.

The Society has recently taken over the stock of the Chicago Papers and Boston Colloquium Lectures, heretofore in the hands of The Macmillan Company. All publications of the

Society, so far as in stock, are now obtainable directly from the main office. The New Haven Colloquium was published by the Yale University Press, and is sold by them.

The following papers were read at this meeting:

(1) Dr. T. H. Gronwall: "A functional equation in the kinetic theory of gases (second paper)."

(2) Dr. T. H. Gronwall: "On the zeros of the functions $P(z)$ and $Q(z)$ associated with the gamma function."

(3) Dr. T. H. Gronwall: "On the distortion in conformal representation."

(4) Dr. C. A. Fischer: "Equations involving the derivatives of a function of a surface."

(5) Professor E. W. Brown: "Note on the problem of three bodies."

(6) Dr. H. Bateman: "A certain system of linear partial differential equations."

(7) Dr. H. Bateman: "On multiple electromagnetic fields."

(8) Mr. A. R. Schweitzer: "On a new representation of a finite group."

(9) Mr. A. R. Schweitzer: "Definition of new categories of functional equations."

(10) Professor E. B. Wilson: "Critical speeds for flat disks in a normal wind: theory."

(11) Professor E. B. Wilson: "A mathematical table that contains chiefly zeros."

(12) Professor E. B. Wilson: "Changing surface to volume integrals."

(13) Dr. T. H. Gronwall: "Elastic stresses in an infinite solid with a spherical cavity."

(14) Dr. T. H. Gronwall: "On the influence of keyways on the stress distribution in cylindrical shafts."

(15) Professor O. E. Glenn: "The formal modular invariant theory of binary quantics."

(16) Professor O. E. Glenn: "The concomitant system of a conic and a bilinear connex."

(17) Dr. P. R. Rider: "Trigonometric functions for extremal triangles."

(18) Mr. H. S. Vandiver: "Symmetric functions of systems of elements in a finite algebra and their connection with Fermat's quotient and Bernoulli's numbers (second paper)."

(19) Mr. S. A. Joffe: "Calculation of eulerian numbers from central differences of zero."

The papers of Dr. Bateman, Mr. Schweitzer, Mr. Vandiver, and the first and third papers of Professor Wilson were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In a previous paper (*Annals of Mathematics*, September, 1915) Dr. Gronwall has shown that, $\varphi(\xi, \eta, \zeta)$ being the logarithm of the function defining the distribution of velocities, the functional equation for φ resulting from the Maxwell-Boltzmann fundamental theorem has as general solution the expression $a + b_1\xi + b_2\eta + b_3\zeta + c(\xi^2 + \eta^2 + \zeta^2)$, under the sole assumption of the continuity of φ for all finite values of the variables. In the present paper, it is shown that this condition may be replaced by the weaker one that φ shall be bounded in an arbitrarily small neighborhood of the origin.

2. Completing a result due to Bourguet, Haskins has recently shown (*Transactions*, 1915) that

$$P(z) = \sum_0^\infty \frac{(-1)^\nu}{\nu!} \frac{1}{z+\nu}$$

has one and only one zero in each of the intervals $-2m - \frac{3}{2} < z < -2m - 1$ and $-2m - 2 < z < -2m - \frac{3}{2}$ $(m = 2, 3, \cdots)$. The proof depends on the Budan-Fourier theorem. In the present paper, Dr. Gronwall shows, by a simpler method, that the real zeros of $P(z)$ lie one in each of the intervals $-2m - 1 - 6/(2m)! \leqq z \leqq -2m - 1 - 1/(2m)!$ and $-2m - 2 + 1/(2m+1)! \leqq z \leqq -2m - 2 + 6/(2m+1)!$, and furthermore that $P(z)$ has exactly four complex zeros, the real parts of which lie between $-\frac{17}{4}$ and $-\frac{3}{2}$, and the imaginary parts between $-\frac{11}{2}$ and $+\frac{11}{2}$. Finally it is shown, by an extremely simple application of Picard's theorem, that the entire function $Q(z) = \Gamma(z) - P(z)$ has an infinity of zeros.*

* It should be noted that Bourguet, *Comptes Rendus*, vol. 96 (1883), pp. 1487–1490, indicated approximations to the real zeros of $P(z)$, which are of essentially the same order as those of Dr. Gronwall. Bourguet also asserted that $P(z)$ can have at most four complex zeros. His proof, however, has been criticized by Nielsen, Handbuch der Theorie der Gammafunktion, p. 36, as inaccurate. It is worth noting, however, that by making use of the known analytic character of $P(z)$, and of the difference equation satisfied by it, Bourguet's proof can be made rigorous, and that then it gives, when taken in connection with Haskins's completion of Bourguet's theorem, the further fact that $P(z)$ has exactly four complex zeros. C. N. HASKINS.

3. In the present paper, Dr. Gronwall gives the exact determination of the bounds in Koebe's distortion theorem, which may now be stated in the following, and final, form:

"When the analytic function

$$w = z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

gives the conformal representation of the circle $|z| < 1$ on a simple region D in the w-plane, then we have for $|z| = r$ and $0 < r < 1$

$$\frac{1-r}{(1+r)^3} < \left|\frac{dw}{dz}\right| < \frac{1+r}{(1-r)^3},$$

$$\frac{r}{(1+r)^2} < | w | < \frac{r}{(1-r)^2},$$

except when $w = z/(1 - e^{\alpha i}z)^2$, in which case the upper and lower bounds are reached for $z = re^{-\alpha i}$ and $z = -re^{-\alpha i}$ respectively. When the region D is convex, we have

$$\frac{1}{(1+r)^2} < \left|\frac{dw}{dz}\right| < \frac{1}{(1-r)^2},$$

$$\frac{r}{1+r} < | w | < \frac{r}{1-r},$$

except for $w = z/(1 - e^{\alpha i}z)$, in which case the upper and lower bounds are reached as before."

It is also shown that the convexity bound, i. e. the upper bound of those values of r for which the map of $|z| < r$ is convex, exceeds $2 - \sqrt{3}$ except when $w = z/(1 - e^{\alpha i}z)^2$, where this value is reached. The above results are then extended in two directions, first by assuming the region D invariant for a rotation of $2\pi/n$ about the origin, and second, by assigning a priori the value of the second coefficient in w. A detailed abstract will appear in the *Comptes rendus*.

4. The functions considered in Dr. Fischer's paper depend on a surface and also on all of the values taken by a given function at points of the surface. Such a function has two partial fonctionnelle derivatives. The condition of integrability of an equation involving these two derivatives is found, and the characteristics are briefly discussed. Similar equa-

tions, involving functions of lines instead of surfaces, are discussed by Lévy in volume 37 of the *Rendiconti del Circolo Matematico di Palermo.*

5. Professor Brown's note gives a form for the equations of motion in the "restricted" case of the problem of three bodies when the first describes an elliptic orbit about the second, and the third is of zero mass. The curves of zero velocity are obtained and some other consequences deduced.

6. Dr. Bateman's first paper appeared in full in the April BULLETIN.

7. A vector field specified by two vectors H and E may be called a multiple electromagnetic field of rank n when the complex vector $M = H + iE$ satisfies the system of partial differential equations

$$\mathrm{rot}_p\, M = -\frac{i}{a}\frac{\partial M}{\partial t_p}, \quad \mathrm{div}_p\, M = 0 \quad (p = 1, 2, \cdots, n).$$

Dr. Bateman shows that each component of M is a right-handed multiple wave function of rank n and obtains various expressions for M by generalizing the solutions of Maxwell's equations which have been given by Hertz, Righi and others. A generalization is also obtained of a theorem due to Appell.

8. In previous communications* Mr. Schweitzer has indicated how any finite group can be represented formally by classes of functional equations derived from the generatrix: $f\{u_1, u_2, \cdots, u_{n+1}\} = \psi(x_1, x_2, \cdots, x_{n+1})$, where $u_i = f_i(t_1{}^{(i)}, t_2{}^{(i)}, \cdots, t_n{}^{(i)}, x_i)$ $(i = 1, 2, \cdots, n+1;\ n = 1, 2, 3, \cdots)$. Perhaps the simplest class of functional equations representing any finite group is the class† $\{E(\begin{smallmatrix}1, & 2, & \cdots, & [n+1] \\ i_1, & i_2, & \cdots, & i_{n+1}\end{smallmatrix})\}$. The author obtains the general solution of this class of equations without referring to any particular substitution, and then proves the following theorem: Given any finite group G, there corresponds to this group a group of functional equations $E(G)$ such that (1) the groups G and $E(G)$ are simply isomorphic, (2) every equation of the group $E(G)$ possesses a solution.

* Cf. BULLETIN, vol. 22, pp. 4, 171.
† For the notation see BULLETIN, l. c., p. 4.

9. The categories of functional equations are defined by Mr. Schweitzer by means of the following generatrices:

$$f(u_1, u_2, \cdots, u_{n+1}) = \psi(x_1, x_2, \cdots, x_{n+1}) \\ + \phi(t_1^{(n+2)}, t_2^{(n+2)}, \cdots, t_n^{(n+2)}),$$

where $u_i = f_i(t_1^{(i)}, t_2^{(i)}, \cdots, t_n^{(i)}, x_i)$ $(i = 1, 2, \cdots, n + 1;$ $n = 1, 2, \cdots)$ or analogous generatrices obtained by applying the following processes, either separately or in combination: (1) interchanging one of the u's with the symbol $\psi(x_1, x_2, \cdots, x_{n+1})$; (2) transposing homologously the x's on the left-hand side. From the preceding generatrices are derived functional equations by replacing some, none, or all of the t's with x's and assuming that the remaining t's (if any) with the same subscript are identical. When the function ϕ is identically zero, the functional equations thus defined include equations defined previously by the author. A simple example of one of the new types of functional equations is the relation

$$\lambda f\{f(t_1, t_2, \cdots, t_n, x_1), \cdots, f(t_1, t_2, \cdots, t_n, x_{n+1})\} \\ = \lambda f\{\alpha_1(x_{i_1}), \alpha_2(x_{i_2}), \cdots, \alpha_{n+1}(x_{i_{n+1}})\} + \phi(t_1, t_2, \cdots, t_n),$$

where the i_k range over $1, 2, \cdots, n + 1$ and are distinct.

10. In calibrating the new wind tunnel at the Massachusetts Institute of Technology, Lieutenant J. C. Hunsaker, U. S. N., instructor in aeronautics, found a critical velocity for disks (2″ to 6″ in diameter) around 10 to 20 miles per hour. By considering the disks as ellipsoids of revolution and the air as a perfect liquid moving irrotationally, Professor Wilson shows that theoretically such a critical velocity, arising from incipient cavitation, should occur at velocities not much greater than those found experimentally. The sharp edges of the disks and the imperfections of the fluid and of its flow would probably lower the theoretical value found. The paper will appear in connection with Lieutenant Hunsaker's in *Smithsonian Miscellaneous Collections,* volume 62, number 4, article X.

11. Statistical results on the distribution of the different digits in the decimal development of an irrational number or in a tabulation of the values of a function appear not to be

numerous. Professor Wilson sets up an infinite series which defines a function that is perfectly healthy from the analytic viewpoint but has this peculiar statistical malady: The irrational value of the function for any terminating decimal value of the argument has in its infinite decimal development only an infinitesimal proportion of digits other than zero. This problem arose incidentally out of conversations with Professor Josiah Royce on probability and statistics.

12. Professor Wilson's note appeared in full in the April BULLETIN.

13. In the present paper, Dr. Gronwall determines the stress distribution in a solid with a spherical cavity, the dimensions of the solid being very large in comparison with the radius of the sphere, and the stress at a large distance from the cavity being assumed as pure tension or compression of constant magnitude T. It is shown that the maximum tensile stress at the surface of the cavity will be

$$\frac{39\lambda + 54\mu}{18\lambda + 28\mu} T,$$

λ and μ being the constants of elasticity. Thus the maximum stress is roughly twice as large as when there is no cavity, which corrects a statement by Larmor (*Philosophical Magazine*, 1892) that the influence of a cavity on the distribution of tensile stresses is negligible.

14. Dr. Gronwall considers a cylindrical shaft, into which is cut a keyway in the shape of a cylinder intersecting the former orthogonally, the shaft being subjected to torsional stress only. This shape of keyway is common enough in practice. The stress distribution is first determined by the methods of the theory of elasticity, and from this exact result an approximate formula is derived, adapted to practical use, and giving the ratio between the maximum stresses in the keyed and the full portions of the shaft as

$$\frac{2}{1 - 3b^2/a^2},$$

a and b being the radii of shaft and keyway respectively.

15. The first part of Professor Glenn's paper deals with methods of constructing formal invariants and covariants. A theorem is given on the determination of a covariant whose leading coefficient is any assigned seminvariant. Fundamental systems modulo 2 of first degree concomitants are derived for the quartic and the quintic.

In the second part it is shown that the higher forms are reducible in terms of their own first degree concomitants modulo 2. It is proved that if the systems of concomitants modulo 2 of the forms of orders 1, 2, 3 are all finite then the system of any form of order m is finite. It is then shown for the binary cubic modulo 2 that every covariant is quasi-reducible, on multiplication by a power of a first degree invariant, in terms of a set of fourteen concomitants.

16. By a ternary transvection process which Professor Glenn described in a paper communicated in February, 1915, he has derived the fundamental system of simultaneous concomitants of the set consisting of a singly quadric and a doubly linear form. The system furnished initially by the method contains above 100 forms. By various reduction processes he has reduced all but 67 forms. These are given as generalized transvectants and in terms of the symbols.

17. The triangles considered in Dr. Rider's paper are formed by extremal arcs. At one vertex the transversality condition of the calculus of variations is satisfied, thus making the triangles correspond to right-angled triangles in euclidean plane geometry. The trigonometric functions are defined as the limits of the ratios of the lengths of the sides as one side approaches zero, where by length is meant the value of a definite integral taken from one extremity of the arc to the other.

18. The present paper consists mainly of extensions along the lines indicated in Mr. Vandiver's first article under the same title (presented to the Society, April, 1913). A conjugate set in a finite algebra A is defined as a system of elements which are reproduced on multiplication of each element by some element of A. Symmetric functions formed by these sets are studied and for the case where A is represented by the incongruent residues modulo m, a rational integer, the

results obtained involve Fermat's quotient and Bernoulli's numbers.

19. In the *Quarterly Journal*, volume 45 (1914), pages 1–51, Professor J. W. L. Glaisher has calculated the first 27 eulerian numbers from certain recurring formulas and has shown that the method was especially advantageous when "curtate" formulas were employed. Mr. Joffe has verified Professor Glaisher's results and has extended the calculations to five more eulerian numbers by a different method based upon the formula

$$E_n = \sum_{m=0}^{n} (-1)^{m+n} e_{m,n},$$

where $e_{m,n}$ denotes $(1/2^m)\delta^{2m}0^{2n}$, and the quantities $\delta^{2m}0^{2n}$ are "central differences of zero." The successive terms $e_{m,n}$ are computed by a continuous process from the recurring formula

$$e_{m,n} = m[me_{m,n-1} + (m + \overline{m-1})e_{m-1,n-1}],$$

and the final values of E_n are verified by the formula

$$E_n = \sum_{m=1}^{n-1} (-1)^{m+n+1}[(m+1)(m+2) - 1]e_{m,n-1}.$$

F. N. Cole,
Secretary.

ON A CONFIGURATION ON CERTAIN SURFACES.

BY PROFESSOR C. H. SISAM.

(Read before the American Mathematical Society, April 21, 1916.)

The surfaces here under consideration are rational and are generated by conics. They may be represented birationally on the plane in such a way that, to the plane or hyperplane sections of a given surface of the given type, correspond curves of order n having in common an $(n-2)$-fold point P_0 and Δ simple points $P_1, P_2, \cdots, P_\Delta$. We suppose further that $\Delta = 2k$, so that the surface is of even order, that $n > 3$ and that $k > 2$. For simplicity, we suppose that the fundamental points $P_0, P_1, P_2, \cdots, P_{2k}$ are in generic position.

The generating conics on the surface are determined by the