modulo p. In counting the number of incongruent fractions in the set (4) we must therefore consider the number of representations (5). We shall regard two representations

$$mn' + m'n = p, \quad m_1n_1' + m_1'n_1 = p$$

as the same if and only if $m = m_1$, $n' = n_1'$, $m' = m_1'$, $n = n_1$. If N is the number of representations of this type, then the relations (6) show that

$$N = K - (p - 1).$$

Now K by definition is equal to twice the number of distinct positive irreducible fractions whose numerators and denominators are each not greater than $\sqrt{p}$. Hence*

$$K = 4(\varphi(2) + \varphi(3) + \cdots + \varphi([\sqrt{p}])) + 2,$$

where $\varphi(k)$ denotes the number of integers $< k$ and prime to it. We therefore have

THEOREM III. *If p is a prime, then the number of representations of p in the form*

$$xy + x'y',$$

where x, y, x', y' are all positive integers $< \sqrt{p}$, is equal to

$$-(1 + p) + 4\sum_{k=1}^{[\sqrt{p}]} \varphi(k).$$

PROOF OF A GENERAL THEOREM ON THE LINEAR DEPENDENCE OF p ANALYTIC FUNCTIONS OF A SINGLE VARIABLE.

BY MR. HAROLD MARSTON MORSE.

(Read before the American Mathematical Society, September 5, 1916.)

A PROOF of the following theorem has to my knowledge not been published to date. The theorem contains as a special case the ordinary theorem concerning the wronskian. Its usefulness in a general treatment of single-valued func-

* Lucas, Théorie des Nombres, p. 393.

tions on a Riemann surface by means of abelian integrals of the second kind was pointed out by Professor Osgood in a lecture course just completed at Harvard.

Let there be given p functions, $f_1(t), f_2(t), \cdots, f_p(t)$, analytic in a region S of the t-plane. Consider the p-square functional determinant, the ith row of which ($i = 1, 2, \cdots, p$) is

$$f_i^{(\lambda_1)}(t_1), f_i^{(\lambda_1-1)}(t_1), \cdots, f_i(t_1), f_i^{(\lambda_2)}(t_2), f_i^{(\lambda_2-1)}(t_2), \cdots, f_i(t_2), \\ \cdots, \cdots, f_i^{(\lambda_\mu)}(t_\mu), f_i^{(\lambda_\mu-1)}(t_\mu), \cdots, f_i(t_\mu).$$

We denote this determinant by $D[t_1, t_2, \cdots, t_\mu]$. We shall have occasion to indicate determinants of the type of D by enclosing the ith row, without the subscript i, in two vertical bars.

THEOREM. *A necessary and sufficient condition for the linear dependence of $f_1, f_2, \cdots, f_p$ is that D vanish identically in all of its arguments.*

Consider a determinant $\bar{D}[\bar{t}, t_1, t_2, \cdots, t_\mu]$, obtained from D by replacing the first column of D by a column $f_1(\bar{t}), f_2(\bar{t}), \cdots, f_p(\bar{t})$, where $\bar{t}$ is a variable independent of $t_1, t_2, \cdots, t_\mu$. We will first show that $\bar{D} \equiv 0$ in all of its arguments, if the same is true for D. If $\lambda_1 = 0$, we have immediately that

$$\bar{D}[\bar{t}, t_1, t_2, \cdots, t_\mu] \equiv D[\bar{t}, t_2, \cdots, t_\mu] \equiv 0.$$

If $\lambda_1 > 0$, we will prove that $\bar{D} \equiv 0$ by showing that at any point $t_1, t_2, \cdots, t_\mu$, and for $\bar{t} = t_1$

$$(1) \quad \frac{\partial^n}{\partial \bar{t}^n}\bar{D} = |f^{(n)}(t_1), f^{(\lambda_1-1)}(t_1), f^{(\lambda_1-2)}(t_1), \cdots, \cdots, \\ f'(t_\mu), f(t_\mu)| = 0 \qquad (n = 1, 2, 3, \cdots),$$

where the last $p - 1$ columns of the determinant of (1) are the same as the corresponding columns of D. Equation (1) holds for $n = 1$, since for $n = 1$ the determinant of (1) either has two columns identical, or else is the determinant D. We proceed to prove by mathematical induction that (1) holds for all values of n. We therefore assume that (1) holds for $n = m$, that is, that

$$(2) \quad |f^{(m)}(t_1), f^{(\lambda_1-1)}(t_1), f^{(\lambda_1-2)}(t_1), \cdots, \cdots, f'(t_\mu), f(t_\mu)| \equiv 0$$

in $t_1, t_2, \cdots, t_\mu$. Upon differentiating (2) with respect to t_1, we have

$$
(3)\quad \begin{aligned} &|f^{(m+1)}(t_1),\ f^{(\lambda_1-1)}(t_1),\ f^{(\lambda_1-2)}(t_1),\ \cdots,\ \cdots,\ f'(t_\mu),\ f(t_\mu)| \\ &\quad + |f^{(m)}(t_1), f^{(\lambda_1)}(t_1), f^{(\lambda_1-2)}(t_1),\ \cdots,\ \cdots, f'(t_\mu), f(t_\mu)| \equiv 0, \end{aligned}
$$

where the last $p-1$ columns of the first determinant of (3) are the same as those of (1), and the last $p-2$ columns of the second determinant of (3) are the same as those of (1).

To prove the second determinant of (3) equal to zero, consider a matrix made up of the columns of D together with the first column of the determinant of (2). We assume for the present that some cofactor $\bar{A}$ of the elements of the first column of D does not vanish identically. We observe that in the matrix above there are only two p-rowed determinants having $\pm \bar{A}$ as a first minor, namely (2) and D. Whence* any p-rowed determinant of the given matrix vanishes. The latter determinant of (3) is such a determinant. Whence the first determinant of (3) vanishes identically. The induction is complete, and we have that

$$\bar{D}[\bar{t}, t_1, t_2, \cdots, t_\mu] \equiv 0.$$

If $\bar{D}$ be developed with respect to the elements of its first column, and for a set of values of $t_1, t_2, \cdots, t_\mu$, for which the above mentioned cofactor $\bar{A} \neq 0$, we have

$$A_1 f_1(\bar{t}) + A_2 f_2(\bar{t}) + \cdots + A_p f_p(\bar{t}) \equiv 0$$

in $\bar{t}$, where the A's are constants with respect to $\bar{t}$, and not all zero. The proof of the theorem is complete under the assumption that some cofactor $\bar{A} \neq 0$.

In the contrary case, we take in place of D, and from the upper right-hand corner of D, the largest determinant for which not all of the cofactors of the elements of its first column vanish identically. There will be such a determinant unless all the f's vanish identically, in which latter case the f's are obviously linearly dependent. Further such a determinant will be of the same general form as D. It will be identically zero in all of its arguments; for if not, it could not be the largest determinant for which not all of the cofactors of

* If in a given matrix a certain r-rowed determinant is not zero, and all the $(r+1)$-rowed determinants of which this r-rowed determinant is a first minor are zero, then all the $(r+1)$-rowed determinants of the matrix are zero. Cf. Bôcher, Introduction to Higher Algebra, p. 54.

the elements of its first column vanish identically. The proof applied to D is therefore applicable to this determinant and the theorem is proved in general.

HARVARD UNIVERSITY,
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NOTE ON THE LINEAR DEPENDENCE OF ANALYTIC FUNCTIONS.

BY DR. G. A. PFEIFFER.

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THE theorem proved in the preceding note, that a necessary and sufficient condition that p analytic functions, $f_1(t)$, $f_2(t)$, $\cdots, f_p(t)$, be linearly dependent is that the determinant whose ith row is

$$f_i^{(\lambda_1)}(t_1),\ f_i^{(\lambda_1-1)}(t_1),\ \cdots,\ f_i(t_1),\ f_i^{(\lambda_2)}(t_2),\ f_i^{(\lambda_2-1)}(t_2),\ \cdots,\ f_i(t_2),$$
$$\cdots,\ \cdots,\ f_i^{(\lambda_\mu)}(t_\mu),\ f_i^{(\lambda_\mu-1)}(t_\mu),\ \cdots,\ f_i(t_\mu)$$

$$(i = 1, 2, \cdots, p;\ p = \mu + \sum_{i=1}^{\mu} \lambda_i)$$

vanish identically in $t_1, t_2, \cdots, t_p$, can be readily proved if we assume the fundamental theorem that the identical vanishing of the wronskian of p analytic functions implies their linear dependence.

By rearranging the columns of the determinant of the theorem we obtain the determinant Δ whose ith row is

$$f_i(t_1),\ f_i'(t_1),\ \cdots,\ f_i^{(\lambda_1)}(t_1),\ f_i(t_2),\ f_i'(t_2),\ \cdots,\ f_i^{(\lambda_2)}(t_2),$$
$$\cdots,\ \cdots,\ f_i(t_\mu),\ f_i'(t_\mu),\ \cdots,\ f_i^{(\lambda_\mu)}(t_\mu).$$

Without losing any generality we shall assume that

$$\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_\mu.$$

Now the derivative of order nq of the q-rowed determinant whose ith row $(i = 1, 2, \cdots, q)$ is

$$f_i(t),\ f_i'(t),\ \cdots,\ f_i^{(q-1)}(t)$$

is equal to a positive integer times the q-rowed determinant whose ith row is