modulo \( p \). In counting the number of incongruent fractions in the set (4) we must therefore consider the number of representations (5). We shall regard two representations

\[ mn' + m'n = p, \quad m_1n_1' + m_1'n_1 = p \]

as the same if and only if \( m = m_1, \ n' = n_1', \ m' = m_1' \), \( n = n_1 \). If \( N \) is the number of representations of this type, then the relations (6) show that

\[ N = K - (p - 1). \]

Now \( K \) by definition is equal to twice the number of distinct positive irreducible fractions whose numerators and denominators are each not greater than \( \sqrt{p} \). Hence*

\[ K = 4(\varphi(2) + \varphi(3) + \cdots + \varphi(\lfloor \sqrt{p} \rfloor)) + 2, \]

where \( \varphi(k) \) denotes the number of integers \( < k \) and prime to it. We therefore have

**THEOREM III.** If \( p \) is a prime, then the number of representations of \( p \) in the form

\[ xy + x'y', \]

where \( x, y, x', y' \) are all positive integers \( < \sqrt{p} \), is equal to

\[ -(1 + p) + 4 \sum_{k=1}^{\lfloor \sqrt{p} \rfloor} \varphi(k). \]

PROOF OF A GENERAL THEOREM ON THE LINEAR DEPENDENCE OF \( p \) ANALYTIC FUNCTIONS OF A SINGLE VARIABLE.

BY MR. HAROLD MARSTON MORSE.

(Read before the American Mathematical Society, September 5, 1916.)

A PROOF of the following theorem has to my knowledge not been published to date. The theorem contains as a special case the ordinary theorem concerning the wronskian. Its usefulness in a general treatment of single-valued func-

* Lucas, Théorie des Nombres, p. 393.
tions on a Riemann surface by means of abelian integrals
of the second kind was pointed out by Professor Osgood in a
lecture course just completed at Harvard.

Let there be given \( p \) functions, \( f_1(t), f_2(t), \ldots, f_p(t) \), analytic
in a region \( S \) of the \( t \)-plane. Consider the \( p \)-square functional
determinant, the \( i \)th row of which \((i = 1, 2, \ldots, p)\) is

\[
f_i^{(0)}(t_1), f_i^{(0-1)}(t_2), \ldots, f_i^{(0)}(t_1), f_i^{(0-1)}(t_2), \ldots, f_i(t_2), \ldots, f_i^{(0)}(t_\mu), f_i^{(0-1)}(t_\mu), \ldots, f_i(t_\mu).
\]

We denote this determinant by \( D[t_1, t_2, \ldots, t_\mu] \). We shall
have occasion to indicate determinants of the type of \( D \) by
enclosing the \( i \)th row, without the subscript \( i \), in two vertical
bars.

**Theorem.** A necessary and sufficient condition for the
linear dependence of \( f_1, f_2, \ldots, f_p \) is that \( D \) vanish identically
in all of its arguments.

Consider a determinant \( \overline{D}[\tilde{t}_1, t_1, t_2, \ldots, t_\mu] \), obtained from
\( D \) by replacing the first column of \( D \) by a column \( f_1(\tilde{t}), f_2(\tilde{t}), \ldots, f_p(\tilde{t}) \), where \( \tilde{t} \) is a variable independent of \( t_1, t_2, \ldots, t_\mu \).

We will first show that \( D \equiv 0 \) in all of its arguments, if the
same is true for \( \overline{D} \).

If \( \lambda_1 = 0 \), we have immediately that

\[
\overline{D}[\tilde{t}, t_1, t_2, \ldots, t_\mu] \equiv D[t, t_2, \ldots, t_\mu] = 0.
\]

If \( \lambda_1 > 0 \), we will prove that \( \overline{D} \equiv 0 \) by showing that at
any point \( t_1, t_2, \ldots, t_\mu \), and for \( \tilde{t} = t_1 \)

\[
\frac{\partial^n}{\partial t^n} \overline{D} = \left| f^{(n)}(t_1), f^{(n-1)}(t_1), f^{(n-2)}(t_1), \ldots, f'(t_\mu), f(t_\mu) \right| = 0 \quad (n = 1, 2, 3, \ldots),
\]

where the last \( p - 1 \) columns of the determinant of (1) are
the same as the corresponding columns of \( D \). Equation (1)
holds for \( n = 1 \), since for \( n = 1 \) the determinant of (1)
either has two columns identical, or else is the determinant \( D \).

We proceed to prove by mathematical induction that (1)
holds for all values of \( n \). We therefore assume that (1) holds
for \( n = m \), that is, that

\[
\left| f^{(m)}(t_1), f^{(m-1)}(t_1), f^{(m-2)}(t_1), \ldots, f'(t_\mu), f(t_\mu) \right| = 0
\]
in \( t_1, t_2, \ldots, t_\mu \). Upon differentiating (2) with respect to \( t_1 \),
we have
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(3) \[ \begin{vmatrix} f^{\prime}(t_1), f^{(\lambda_1-1)}(t_1), f^{(\lambda_1-2)}(t_1), \ldots, f'(t_\mu), f(t_\mu) \\ & + \begin{vmatrix} f^{(m)}(t_1), f^{(\lambda_1)}(t_1), f^{(\lambda_1-2)}(t_1), \ldots, f'(t_\mu), f(t_\mu) \end{vmatrix} \equiv 0, \]

where the last \( p - 1 \) columns of the first determinant of (3) are the same as those of (1), and the last \( p - 2 \) columns of the second determinant of (3) are the same as those of (1).

To prove the second determinant of (3) equal to zero, consider a matrix made up of the columns of \( D \) together with the first column of the determinant of (2). We assume for the present that some cofactor \( A \) of the elements of the first column of \( D \) does not vanish identically. We observe that in the matrix above there are only two \( p \)-rowed determinants having \( \pm A \) as a first minor, namely (2) and \( D \). Whence* any \( p \)-rowed determinant of the given matrix vanishes. The latter determinant of (3) is such a determinant. Whence the first determinant of (3) vanishes identically. The induction is complete, and we have that

\[ D[\tilde{t}, t_1, t_2, \ldots, t_\mu] \equiv 0. \]

If \( \tilde{D} \) be developed with respect to the elements of its first column, and for a set of values of \( t_1, t_2, \ldots, t_\mu \), for which the above mentioned cofactor \( \tilde{A} \neq 0 \), we have

\[ A_1f_1(\tilde{t}) + A_2f_2(\tilde{t}) + \cdots + A_{\mu}f_{\mu}(\tilde{t}) \equiv 0 \]

in \( \tilde{t} \), where the \( A \)'s are constants with respect to \( \tilde{t} \), and not all zero. The proof of the theorem is complete under the assumption that some cofactor \( \tilde{A} \neq 0 \).

In the contrary case, we take in place of \( D \), and from the upper right-hand corner of \( D \), the largest determinant for which not all of the cofactors of the elements of its first column vanish identically. There will be such a determinant unless all the \( f \)'s vanish identically, in which latter case the \( f \)'s are obviously linearly dependent. Further such a determinant will be of the same general form as \( D \). It will be identically zero in all of its arguments; for if not, it could not be the largest determinant for which not all of the cofactors of

* If in a given matrix a certain \( r \)-rowed determinant is not zero, and all the \( (r + 1) \)-rowed determinants of which this \( r \)-rowed determinant is a first minor are zero, then all the \( (r + 1) \)-rowed determinants of the matrix are zero. Cf. Bôcher, Introduction to Higher Algebra, p. 54.
the elements of its first column vanish identically. The
disproof applied to $D$ is therefore applicable to this determinant
and the theorem is proved in general.

Harvard University,
June 1, 1916.

NOTE ON THE LINEAR DEPENDENCE OF
ANALYTIC FUNCTIONS.

By Dr. G. A. Pfeiffer.

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The theorem proved in the preceding note, that a necessary
and sufficient condition that $p$ analytic functions, $f_1(t), f_2(t), \ldots, f_p(t)$, be linearly dependent is that the determinant whose
$i$th row is

$$f_i^{(\lambda_1)}(t_1), f_i^{(\lambda_2-1)}(t_2), \ldots, f_i^{(\lambda_q)}(t_2), f_i^{(\lambda_2-1)}(t_2), \ldots, f_i(t_2), \ldots, f_i^{(\lambda_q)}(t_\mu), f_i^{(\lambda_2-1)}(t_\mu), \ldots, f_i(t_\mu)$$

($i = 1, 2, \ldots, p$; $p = \mu + \sum \lambda_i$)

vanish identically in $t_1, t_2, \ldots, t_p$, can be readily proved if we
assume the fundamental theorem that the identical vanishing
of the wronskian of $p$ analytic functions implies their linear
dependence.

By rearranging the columns of the determinant of the
theorem we obtain the determinant $\Delta$ whose $i$th row is

$$f_i(t_1), f_i'(t_1), \ldots, f_i^{(\lambda_1)}(t_1), f_i(t_2), f_i'(t_2), \ldots, f_i^{(\lambda_2-1)}(t_2), \ldots, f_i(t_\mu), f_i'(t_\mu), \ldots, f_i^{(\lambda_q)}(t_\mu).$$

Without losing any generality we shall assume that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\mu.$$

Now the derivative of order $nq$ of the $q$-rowed determinant
whose $i$th row ($i = 1, 2, \ldots, q$) is

$$f_i(t), f_i'(t), \ldots, f_i^{(q-1)}(t)$$

is equal to a positive integer times the $q$-rowed determinant
whose $i$th row is