$$
(9) \quad \begin{aligned}
&(1, 3, 4) = -w_2w_4/w_1^2, && (1, 3, 5) = 0,\\
&(1, 3, 6) = w_2^2/w_1^2, && (1, 4, 5) = w_3w_4/w_1^2,\\
&(1, 4, 6) = -w_2/w_1, && (1, 5, 6) = w_2w_3/w_1^2,\\
&(2, 3, 4) = w_4/w_1, && (2, 3, 5) = -w_3w_4/w_1^2,\\
&(2, 3, 6) = -w_2/w_1, && (2, 4, 5) = 0,\\
&(2, 4, 6) = 1, && (2, 5, 6) = -w_3/w_1,\\
&(3, 4, 5) = w_4^2/w_1^2, && (3, 4, 6) = 0,\\
&(3, 5, 6) = w_2w_4/w_1^2, && (4, 5, 6) = -w_4/w_1.
\end{aligned}
$$

Since the non-vanishing fractions in $w_1, \cdots, w_4$ all have second order numerators and a common denominator w_1^2, the theorem is proved. Substitution of these results in (8) and the results from (8) in I gives the explicit form of the translation surface $\varphi(w)$, in a form free from extraneous factors.

It is obvious that a complete set of invariants gives, in the present case of the congruence (m, n) or in the previous *binary* case of $(m, 1)$, a fundamental system of translation surfaces. For the congruence (2, 2), cut by a plane in a quadrilateral, the complete system consists of four surfaces $(aa'a'')^2$, $(bb'b'')^2$, $(aa'b)^2$, $(abb')^2$, all of degree 3 and class 4.

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PAPPUS. INTRODUCTORY PAPER.

BY DR. J. H. WEAVER.

(Read before the American Mathematical Society, April 24, 1915.)

ONE of the most wholesome tendencies in the study of mathematics today is the desire to give increased attention to the history and genesis of the subject. This tendency has led to a more careful study of the works of the old Greek mathematicians. Of these Pappus of Alexandria was among the last, and from the point of view of the historian one of the most important because it is in his works that we have the only authentic account of the lost works of a large number of preceding mathematicians.

Of the life of Pappus we know practically nothing. Even the date of his activity is known only within the compass of about a century. Suidas places him as a contemporary of Theon of Alexandria (379–95 A.D.), while a tenth century manuscript has a marginal note connecting him with the reign of Diocletian (284–305 A.D.).*

As a writer, Pappus must have been quite versatile if the following list of works attributed to him is any indication:

(1) Description of the World. (2) Comments on the Four Books of the Almagest.† (3) Interpretation of Dreams. (4) On the Rivers of Libya. (5) Commentary on the Analemma of Diodorus.‡ (6) Comments on Euclid's Elements. (7) Comments on Ptolemy's Harmonica. (8) Collection.

Of all these the only one extant even in part is the Collection, which is a summary in eight books of the principal works of preceding Greek mathematicians with comments and lemmas on the works in question. A brief outline of the contents of the Collection is as follows:

Books I and II probably contained an account of the arithmetic of the Greeks. However, all of Book I and part of Book II have been lost. The portion of Book II that remains discusses the methods of multiplication used by Apollonius of Perga.

Book III takes up the geometric side of mathematics. It consists of four parts.§

1. Discussion of the problem of inserting two mean proportionals between two given lines, to which form Hippocrates had reduced the duplication problem.

2. Development of the ten means in use among the Greeks.‖

* In this connection see article on Pappus by Sir Thomas L. Heath, Encyclopedia Britannica, 11th ed.

† This title is given by Suidas, but he is in error on this point, because there are thirteen books instead of four in the Almagest of Ptolemy.

‡ Of the Analemma we know nothing. Pappus is the only writer, so far as I know, who mentions it. See Pappus Alexandrinus Collectio, ed. Hultsch, Berlin, 1876–8, p. 246. Hereafter we will refer to this work as "Pappus."

§ Heath, in the article mentioned in the first note above, gives five divisions, the fifth being a second discussion of the duplication problem. I am classifying these two parts as one.

‖ The means signify the arithmetic, geometric, harmonic, and seven other means closely allied to them. For a discussion of these see article by J. S. Mackay, "Pappus on the progressions," *Proc. Edinburgh Math. Society*, vol. 6, p. 48.

3. Some theorems on the sums of lines drawn from points on the base of a triangle and included by the sides of the triangle, and the extension of these to polygons.

4. Inscription of the five platonic bodies in a sphere.

Book IV contains:

1. Some miscellaneous theorems, most of which have a bearing on tangent circles.

2. A discussion of the properties of the conchoid, the quadratrix and the spiral of Archimedes and their applications to the three famous problems of Greek geometry.

Book V deals with the theory of isoperimetric figures both plane and solid. It consists of three parts:

1. Theory of plane isoperimetric figures.

2. Theory of solids as developed by Archimedes.

3. Comparison of the five platonic bodies when they have equal surfaces.

Book VI discusses some of the more difficult theorems in the minor works on astronomy.*

Book VII is probably the most important one of the eight, for here we have set forth in a careful systematic fashion the fundamental ideas of the Greeks on loci. Just what the contents of this book cover can be summed up in the following paragraph from the introduction.

"This is the order of the books on loci. One book of the Data of Euclid, two books of Proportional Section, two of Spatial Section, two of Determinate Section and two on Contacts, all by Apollonius, three books of the Porisms of Euclid, two books on Inclinations by Apollonius,† two books on Plane Loci and eight on Conics by the same author, five books of Solid Loci by Aristæus, two books of Surface Loci‡ by Euclid and two books on Means by Eratosthenes. There are then in all thirty-three books of which, as far as the conics of Apollonius, I have set forth the contents for your inspection and I have stated the number of loci, of determinations and

* The Almagest of Ptolemy was known as the major work on astronomy and all others were classed as minor. Of these works Pappus gives lemmas on the Spherics of Theodosius, the Moving Sphere of Autolycus, Days and Nights by Theodosius, Size and Distances of the Sun and Moon by Aristarchus, the Optics of Euclid, and the Phænomena of Euclid.

† For a discussion of the term νεῦσις or inclinations see Works of Archimedes, ed. Heath, Cambridge, 1897, Chap. V.

‡ Whether this work discussed surfaces of the second degree only, or sections of these surfaces also is not definitely known. For a discussion of this point see Heiberg, Studien über Euklid, Leipzig, 1882, pp. 79–83.

cases in each book and have set up quite a few lemmas which are required, nor do I think I have omitted any question in the discussion of these books."*

Book VIII deals with some mechanical problems and a few theorems of pure geometry.†

Although we are not sure what theorems were discovered by Pappus, the following are a few which may be due to his genius:

1. Generalization of the pythagorean theorem.‡

2. Several of the fundamental theorems relating to perspectivity.§

3. A theorem showing the relation between the quadratrix and the spiral of Archimedes.

4. Description of a spiral on a sphere and the quadrature of the area of the surface included between this spiral and a great circle of the sphere.||

5. Statement of the so-called theorems of Guldin. These are found without proof in the introduction to book VII. Their statement there is as follows: "Figures generated by a complete rotation about an axis have a ratio compounded from the rotating figures and lines similarly drawn to the axes from the centers of gravity of the rotating figures: and figures generated by an incomplete revolution have a ratio compounded from the rotating figures and from the arcs which the centers of gravity of the rotating figures describe."¶

6. Proof of the invariancy of the cross ratio under a projective transformation.

The best edition of the Collection is that of Hultsch** which is in Greek with a parallel Latin version. Sir Thomas L.

* Pappus, p. 636.

† For a further discussion of the contents of the Collection see the article of Heath mentioned in the first footnote on p. 128. Also see Account of the Life and Writings of Robert Simson, by W. Trail, Bath, 1812, pp. 131–180, and Hutton's Philosophical and Mathematical Dictionary, 2d ed., vol. II, London, 1815, pp. 146–48.

‡ For a statement and proof of this theorem see Teaching of Geometry, D. E. Smith, Boston, 1911, p. 263.

§ See in this connection articles by R. C. Archibald, "Centers of similitude of circles and certain theorems attributed to Monge. Were they known to the Greeks?" *Amer. Math. Monthly*, vol. 22, pp. 6–12, and "Historical note on centers of similitude of circles," *Amer. Math. Monthly*, vol. 23, pp. 159–61.

|| For a discussion of theorems 3 and 4 see Chasles, Aperçu Historique, Bruxelles, 1837, pp. 28 and ff.

¶ Pappus, p. 682.

** See the third footnote on p. 128.

Heath did intend to bring out an English edition, but a recent private letter from Heath speaks for itself: "I do not think that there is much prospect now of my bringing out any book on Pappus, at all events for a long time . . . if I ever have the time to do it at all." The writer has spent the last fifteen months making a careful translation of the Collection, during which time he has noticed that most writers on Pappus have been either extensive or intensive but not both at the same time. In particular he has noticed that most writers on Pappus seem to think that he was guilty of appropriating work that did not belong to him. However, the facts which have come out of this extended study seem to set at naught all such accusations.

First let us turn to a problem that has been most widely discussed, the trisection of an angle. Gow* says that in Book IV Pappus claims as his own a solution of this problem which is doubtless the one that Proclus ascribes to Nicomedes. The solution in question is Prop. 32, which reads as follows: "Let there be any acute angle ABC and let any line AC be drawn perpendicular to BC, and let the rectangle $ACBZ$ be completed, and let ZA be extended to E, which is so assumed that BE being drawn, the segment of it cut off between the lines AC and AE is double the segment AB. Then angle $EBC = 1/3$ angle ABC."

Cantor seems to have some doubts in his mind as to whether this was the solution of Nicomedes effected by means of the conchoid, but does not produce any evidence to support his doubts.†

Hoppe criticizes Cantor for even considering that this solution was the one of Nicomedes and supports his criticisms by pointing out the fact that the insertion of the line DE was accomplished by means of the intersection of a hyperbola and a circle, and that in this connection no mention is made of the conchoid.‡

R. C. Archibald states that the discovery of the application of the conchoid to the trisection problem was claimed by Pappus.§ Sturm says that Pappus seems to claim for him-

* Short History of Greek Geometry, Cambridge, 1884, p. 310.

† M. Cantor, Vorlesungen über Geschichte der Mathematik, 3d ed., Leipzig, 1907, vol. I, p. 354.

‡ Hoppe, Mathematik und Astronomie im klassischen Altertum, Heidelberg, 1911, p. 308.

§ *Amer. Math. Monthly*, vol. 21 (1914), p. 347.

self this application.* The only reference that is cited from the works of Pappus by the above authorities is the following: "We (Pappus) in the Commentary on the Analemma of Diodorus used this curve to trisect an angle."† And from this statement they draw the conclusion that Pappus claims to be the first to apply the conchoid to the trisection problem. But Pappus was not the first to make the application, and does not even claim to be the first, as the following quotation clearly shows. "Nicomedes solved this problem (duplication problem) by means of the conchoid, by means of which he also trisected an angle."‡ This quotation disposes of the question of the claims of Pappus and confirms the statement of Proclus that Nicomedes used the curve to solve both the duplication and trisection problems.§

Also Pappus is accused of proving Prop. 10, Book VIII, in a manner only accidentally different from that of Heron of Alexandria and that he here assumes credit that rightfully belongs to Heron.|| To meet this accusation a translation of the introduction to the proposition in question will be sufficient. It is as follows: "To the same kind of reasoning¶ belongs the problem that a given weight may be moved by a given power. This is found in the Mechanics of Archimedes, in which he is led to say exultingly: 'Give me where I can stand and I will move the world.' Then Heron of Alexandria has clearly explained this in his book called *βαρουλκός* . . . this being assumed, that the diameter of the wheel shall have to the diameter of the axle the ratio 5 : 1, the weight to be moved shall be 1,000 talents and the power which moves it 5 talents. Now let it be proved by us in the ratio 2 : 1 and let the weight be 160 talents and the power 4 talents."**

In the above two instances the evidence is clear that Pappus was not guilty of appropriating work that belonged to others,

* A. Sturm, Das Delische Problem, Linz, 1895–7, p. 80.

† Pappus, p. 246.

‡ Pappus, p. 56.

§ Proclus, ed. Friedlein, p. 272.

|| See Gow, Short History, p. 310, Cajori, History of Elementary Mathematics, New York, 1914, p. 86, and Ball, Short History of Mathematics, 2d ed., New York, 1893, p. 101. Cajori and Ball do not mention this case specifically, but since it is one of the three current accusations I am assuming that they had this instance in mind.

¶ The reasoning referred to is the reasoning in Prop. 9, in which Pappus calculates in general terms the power that will be required to move a weight in a horizontal direction on an inclined plane.

** Pappus, p. 1060.

but in the third case there is no direct evidence in the works of Pappus himself. However, he is accused of claiming as his own fourteen theorems on isoperimetric figures which Theon of Alexandria says were found in a book written by Zenodorus.* Nokk even goes so far as to claim that all the theorems recorded by Pappus on isoperimetric figures were given by Zenodorus in his book on the subject. Hoppe criticizes Nokk very severely and takes the opposite extreme by stating that Pappus probably knew nothing about Zenodorus and worked out the theorems independently.†

The proofs of the theorems mentioned by Theon are almost identical with the corresponding ones in Pappus. This fact along with some other evidence leads Hultsch to think that Theon borrowed part of his commentary from Pappus.‡

But aside from the above criticisms the following items have a bearing on the situation.

(1) The object of the Collection was to set forth the theories that had been previously developed and not to develop a new theory.

(2) There is a possibility that Pappus may have mentioned the name of Zenodorus, for considerable mutilation has taken place in Book V. Proposition 9 has been lost entirely and Proposition 7 has a portion of its proof incorrectly written as we now have it.

(3) In comparing Apollonius and Euclid Pappus states that it is necessary to be kindly disposed toward any one who can advance mathematical theory even a little.§

(4) In the light of the above discussion this is the only instance in the entire Collection as to which there is any doubt about the appropriation of material, and the omission of a single name (if it was originally omitted) is, in a work the size of the Collection, hardly cause enough for an accusation of theft.

These things seem to the writer to be sufficient to give Pappus the benefit of the doubt, and to clear his name from the stigma that has clung to it for so long. At least an accusation of theft seems hardly justifiable until we have some more evidence on the subject.

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* See in this connection Pappus, pp. 1189–1211, Gow, Short History, pp. 271 and 310, Cajori, History of Elementary Mathematics, p. 86, and Nokk, Abhandlung über Isoperimetrische Figuren, Progr., Freiburg, 1860.

† Mathematik und Astronomie, p. 313.

‡ Pappus, vol. III, p. xv.

§ Pappus, p. 676.

On the other hand the following is a notable instance where Pappus is given credit in some quarters for a thing that he does not even claim as his own. Gow* and Taylor† attribute the discovery of the focus and directrix definition for the conic sections to Pappus. Ball qualifies his statement by saying that the discovery is due to Pappus if it was not stolen by him.‡

Cantor and Chasles mention the property but make no statement other than that it is not found in Apollonius.§ Heath states that Pappus gives us the earliest recorded use and proofs of the properties in question but that the discovery was not due to Pappus, || while Zeuthen seems to think that the focus at least for the parabola was known to Euclid.¶

In the Collection there are two references to the properties mentioned above. The first is Prop. 34, Book IV, which reads as follows: "*Some* by other reasoning have explained the trisection of an angle without inclinations.** Let the arc be cut in any ratio for there is no difference whether we cut an arc or an angle. Let it be done and let arc $BC = 1/3$ arc AC and let AB, AC and BC be drawn. Then $\angle ACB = 2\angle BAC$. Let CD bisect $\angle ACB$ and let the lines DE and BZ be drawn perpendicular to AC. Then $AD = DC$ and $AE = EC$. Therefore the point E is given."

Then by means of a very simple proof we have the following relation

$$BZ^2 + ZC^2 = 4EZ^2,$$

from which Pappus draws the following conclusion.

"Now because the points E and C are given and the perpendicular BZ is drawn and given and the ratio $EZ^2 : (BZ^2 + ZC^2)$ is given the point B is on a hyperbola."

* Short History, pp. 252 and 309.

† C. Taylor, Introduction to the Ancient and Modern Geometry of Conics, Cambridge, 1881, p. xxxvi.

‡ Short History, 2d ed., New York, 1893, p. 101.

§ Cantor, Geschichte, vol. I, p. 452, and Chasles, Aperçu Historique, p. 44.

|| See the first footnote on p. 128.

¶ Geschichte der Mathematik im Alterthum und Mittelalter, Kopenhagen, 1896, p. 211. See also in this connection Zeuthen, Die Lehre von den Kegelschnitten im Alterthum, Kopenhagen, 1886, pp. 212 and ff., where an extended account of the properties is found, and Encyclopädie der Mathematischen Wissenschaften, Band III, Heft I, pp. 12 and 52–59. This last gives extensive references relative to the development of the focus and directrix.

** See the third footnote on p. 129.

Now according to the introductory statement of the theorem Pappus was only recording the solution of others, but we have here everything concerning the focus and directrix definition except the definite statement of the theorem, and since Pappus by means of the constant ratio was able to say that the point B was on a hyperbola the distinction must have been known.

In the second reference we have the explicit statement and proof of the properties.*

Proposition 238, Book VII, reads as follows: "Let there be a line AB and a point C in the same plane and from a point D let DC be drawn and also DE perpendicular to AB and let the ratio $CD : DE$ be given. Then the point D is on a conic, and if the ratio $= 1$ it is on a parabola, if > 1 on a hyperbola and if < 1 an ellipse."

The proof of this theorem depends upon the proof that the ratio $EZ^2 : (BZ^2 + ZC^2)$ of Prop. 34, Book IV, is given and known. But this is the theorem that Pappus assumes to be known. This consideration along with the fact that Prop. 238, Book VII, is a lemma on Euclid's Surface Loci gives some weight to the statement of Zeuthen mentioned above. This much is evidently true. The idea was not new to Pappus even if we concede that he was the first to put it in definite form.

One thing more ought to be emphasized in any discussion relative to the Collection and that is its remarkable suggestiveness. In order to understand this, one has only to turn to the works of such men as Chasles,† Günther,‡ Descartes,§ Newton,|| and Steiner,¶ for in the writings of these men it has furnished the basic ideas for analytic geometry, projective geometry, and other allied theories. And if it had so much to offer these men, it ought to furnish some suggestions to the careful reader of today.

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* Pappus, pp. 1005–1015.

† Aperçu Historique, pp. 28 and ff.

‡ "Ueber eine merkwürdige Beziehung zwischen Pappus und Kepler," Biblioteca Mathematica, 1888, p. 81.

§ La Géometrie de René Descartes, Paris, 1886.

|| See Ball, Short History, 2d ed., p. 355.

¶ Steiner, Werke, herausgegeben von Weierstrass, pp. 81 and ff.