with considerable historical matter; Weidler, De Characteribus Numerorum (Wittembergæ, s.a.); Biering, Historia Problematis Cubi duplicandi (Copenhagen, 1844); Brugsch, Numerorum apud veteres Ægyptios Demoticorum Doctrina (Berlin, 1849); and Budaeus, De Asse et Partibus eius Libri V (Paris, 1514, and various other editions), works of no great value, except the last one, but still important enough to make it worth the while of students to consult them.

Such a list of omissions could, of course, be greatly amplified, and no doubt M. Eneström will see that this is done when the publication of *Bibliotheca Mathematica* is resumed. It would be possible also to mention several unimportant misprints, but this matter, too, may well be left for the careful if sometimes caustic pen of the Stockholm critic. The omissions are not mentioned by way of criticism, because no book of this size can be expected to give more than a limited selection from all the works upon the subject, even from the rather important ones; but they are given simply for the purpose of calling attention to the fact that students must not feel that the list given by Professor Loria is exhaustive. The book is merely a guide which points the general way, and the student must expect to supplement it at every step of his progress. Looked at in such a spirit, the book is a very welcome addition to our literature.

DAVID EUGENE SMITH.

Fundamental Conceptions of Modern Mathematics—Variables and Quantities. By ROBERT P. RICHARDSON and EDWARD H. LANDIS. The Open Court Publishing Company, Chicago and London, 1916.

THE volume under review contains Part I: on Variables and Quantities, and a portion of Part X: on Functional Relations, of the division on Algebraic Mathematics of the treatise entitled Fundamental Conceptions of Modern Mathematics. No mention is made of other divisions of the treatise, but twelve additional parts of the first division are announced. A synopsis of these later parts is given at the end of the present volume and the authors invite "suggestions toward improving the present redaction of these later parts" as well as "comments in criticism of Part I." Some of the topics which are to be treated in the later parts are domains and ranges; limits, bounds, and appanages; symbols, signs, and sigla; differentiation; integration, etc.

The purpose of the work is "to examine critically the fundamental conceptions of mathematics as embodied in the current definitions." The authors complain that "Definitions are laid down only as they are needed for the work in hand, and in their formulation attention is given, not to the needs of mathematical science as a whole, but to those of a single book—too often a book whose sole purpose is to enable more or less stupid youths to pose as graduates of a course in mathematics." One feels that they are inspired by uplift ideals when one reads that "mathematics to-day is indeed far behind most other sciences as regards lucidity of exposition. In a comparatively short time a young man of average ability can become so familiar with chemistry or botany or zoology, as to be able to read intelligently a work in any department of the science whatsoever. But this is not the case with mathematics—a student far above mediocrity, who has taken the best university course in mathematics to be found, will come across mathematical works as unintelligible to him as Chinese or Choctaw. It is not merely that he finds himself unfamiliar with the theorems proven in such works: this would be neither surprising nor detrimental; but he will not even be able to understand what it is that the theorems are about. And to gain the knowledge requisite for this will not be a matter of consulting a lexicon; but one of hard study for several months. This state of affairs is not, we hold, an unavoidable one due to the peculiar difficulties of mathematics. It is due to the lack of systematization; and in particular to the failure of text-books to give any thorough exposition of the fundamental conceptions of mathematics." It behooves the mathematician to take notice of such a thesis and to examine sympathetically any contribution made by its positor. The statements that "the thirst for so-called 'original research,' and the credit attached to it has led mathematicians to disregard such matters" and that "in many quarters the impression prevails that there is nothing more to be done at the foundations of mathematics" raise some doubt as to the competence of the authors to speak authoritatively and put the mathematical reader on his guard. He is likely to maintain this attitude when he reads that "This much-needed revision of mathematics ought undoubtedly to be made from a philosophical standpoint, there being constantly maintained rigid adherence to the requirements of a sane metaphysics in the best sense of the word and to the canons of a sound logic" and when he is shocked by the bad taste of the statement that "in making a provisional list" of the transformations of equations, "we find that the treatment of Vieta was more truly scientific than that of the pygmies who followed him in this field."

The importance of Part I with reference to the work as a whole is made clear by the statement that "the keynote of our work is the distinction we find it necessary to make between quantities, values and variables on the one hand, and between symbols and the quantities or variables they denote or values they represent on the other." Here the authors complain that "mathematicians confuse values and quantities, and again quantities and variables, though not usually values and variables. And they also confuse symbols (and in general expressions) with the things these denote or represent" and that "the tendency to confusion instead of distinction would indeed seem to be growing, and certain mathematicians would avowedly make mathematics entirely a matter of symbolism."

The first 145 pages are devoted to a critique of the definition of a variable as a quantity or as a symbol of some sort. An explicit statement of what the authors consider to be the "proper definition" of a variable is nowhere given: "Any attempt to give a precise account of the definition of the term 'variable' would require a somewhat lengthy consideration of the philosophical theory of the categories, which cannot be given in this place." An idea of what they have in mind may be gained perhaps from the following quotations: "The quantities contemplated together, when a variable is the object of inquiry, compose a class of quantities which we may call a variable-class. But a wide gulf separates the inquiries instituted with respect to variables and all other inquiries instituted with respect to the members of classes composed of quantities;" "the propositions enunciated concerning a variable do not, in the typical cases, treat of the members of the variable-class taken separately; they treat of the mutual relations between the members of the class. And among these relations, the most common is that state of affairs which exists when the variable possesses a limit—a limit being a quantity which may or may not belong to the variable;" "in any inquiry concerning a variable, one of the most important, though one of the least regarded facts, is the arrangement in order of the quantities which compose the variable;" "we shall say that the quantities of a variable are arranged in order, when every quantity of the variable has had conferred upon it a relation or order with respect to at least one other quantity of the variable;" "to discover what character quantities must possess to be amenable to those inquiries for which a variable is formed, and therefore to be eligible to join in forming a variable, we will inspect the most typical of all of the relations between the quantities of a variable—the relations which subsist when the variable incessantly approaches a quantity, either as a limit or otherwise. The qualifications hereby educed are conformed to by the quantities of every variable, as an exhaustive examination of the variables found in the mathematical sciences will show. For a variable x to incessantly approach a quantity a, it is requisite: first, that there should be a unifarious arrangement of the quantities of the variable; second, that each quantity in the variable should be *nearer* a than is every previous quantity; third, that no quantity of the variable should be equal to or identical with a. The second condition is that relevant to our inquiry. In stating it we use the word *nearer* in a technical algebraic sense. Of two quantities x_1 and x_2, the latter is said to be *nearer* the quantity a than is the former when the difference between x_2 and a is less numerically than the difference between x_1 and a;" "hence, in order that the variable x shall incessantly approach the quantity a, it must be possible to find the difference between every quantity of x and a—in other words, with each quantity of x, it must be possible to either subtract this quantity from a or to subtract the latter from the former. And for these operations of subtraction to be possible, there must be a certain uniformity of character of a and the quantities of x." We shall not indulge in comment upon these statements. Much space is devoted to a classification of quantities into sorts, kinds and varieties, but unfortunately, we do not find anywhere an explicit statement of what "quantity" means. The authors present their views upon vectors and quaternions (comment on these views will doubtless come from specialists in the field) and proceed to build up a theory of negative, imaginary, complex, and hypercomplex numbers. We are told that "of the services rendered by Hamilton to mathematical science, one of the most important has not been recognized by mathematicians. With the admission of the relations between vectors of the same sort to membership among the quantities of mathematics, there is furnished ample argument to banish forever, to the limbo of doctrines outworn, the tenet so widely taught by mathematicians, even at the present day, that negative, imaginary, and complex 'numbers' are mere symbols." Indeed they hold that "the ordinary algebra of the present day . . . ought to be largely developed as a complanar vector analysis; and it is on such a basis that we deal, not merely with the imaginary quantities but also with the real negatives, the abstract as well as the applicate. The method of introducing the negative real abstract quantities by the sanction of the Principle of Permanence we are constrained to regard as especially unsatisfactory, though this method is used, in formulating algebra as a system, by the most eminent mathematicians. Our own way of dealing with these and with the imaginary abstract quantities is the natural result of not confining our attention to the formal side of algebraic science, but taking into account its matter as well as its form. Those who would develop algebra as a purely formal science, and as nothing more, are satisfied to stop when they have ascribed the origin of the conception of a negative real abstract quantity to such equations as $x + 1 = 0$, $x + 2 = 0$, etc., and the conception of an imaginary abstract quantity to such equations as $x^2 + 1 = 0$, $x^2 + 2 = 0$, etc. But the conceptions attained when one does not look beyond the equations are nothing more than purely formal conceptions of quantities, and he who would master the matter as well as the form of the science must look deeper. He must attain what might be termed *entitative conceptions*. He must find classes of entities adequate to fulfill the conditions fixed by the formal conceptions."

In the next ten pages the authors review and criticize most unmercifully the definitions of a variable given by various authors, including Baire, Gennochi-Peano, Czuber, Burkhardt, Pringsheim, Pierpont, and Russell. Part I closes with a consideration of certain variables. "The most simple of all variables are the ordinary progressions of arithmetic: arithmetical, geometrical, harmonical, etc." Series are considered in the same connection.

The portion of Part X which occupies the last 30 pages of the volume begins with the statement that "the essential characteristic of a functional relation between variables we hold to be the like order of corresponding quantities in these variables. For two variables, y and x, to be in functional relation, it is necessary and sufficient that there be two or more quantities of x, x_1, x_2, etc., which respectively correspond to y_1, y_2, etc., quantities of y, and that with every two pairs of corresponding quantities x_m and y_m, x_n and y_n, y_m is subsequent to y_n when x_m is subsequent to x_n and vice versa; y_m is prev ous to y_n when x_m is previous to x_n and vice versa; and finally when x_m is neither previous nor subsequent to x_n (*e. g.*, is abreast of it, as may be the case under a multifarious arrangement) y_m bears a like relation of order to y_n, and is neither previous nor subsequent to the latter and vice versa. In the case of three or more variables, x, y, z, etc., the sufficient and necessary conditions are quite analogous." Comment seems quite superfluous here.—After reviewing some of the historical definitions of the notion of function, the authors take up current views of functions by quoting from Dini, Harkness and Morley, Osgood, and Pringsheim, none of which authors succeed in finding favor with our crusaders. The last six pages of the volume are devoted to an exposition of the "errors of Riemann."

While some of the criticisms made by the writers of this book are doubtless well-founded (confusion in the use of the words *same, equal,* and *identical;* in the use of the word *series;* apparently erroneous ascription to Dirichlet of the definition of function usually credited to him), and while some of the ideas they introduce are interesting and perhaps valuable, (confluence and contrafluence, e. g.), it must be clear from the passages quoted that the book does not contribute in any important way to the solution of the problem which it set, viz., the critical examination of the fundamental conceptions of mathematics.

In the first place much space is devoted to dealing in a superficial way with the nomenclature of mathematics, and introducing without reason which seems sufficient to the reviewer, a considerable number of new terms. It is not at all clear for instance, why "negative protomonic abstract non-zero" is less "crude" and less "unsuited to the present state of mathematics" than "negative number." While uniformity in usage is doubtless desirable, it is really not of much consequence whether the term "complex number" shall or shall not include the reals or the pure imaginaries or both. To make this issue the basis for a tedious quarter of a page and for the statement that "inability to use language with precision seems to be a failing endemic among mathematicians, and Riemann was not immune" under a page heading "the errors of Riemann" is amusing arrogance. The book does not lack other passages of a similar character. It does not seem necessary nor perhaps even desirable to the reviewer that an elementary text in algebra (as Bauer's Vorlesungen über Algebra or Burnside and Panton's Theory of Equations) should introduce the most general notion of function. The authors seem to find a failure to do so sufficient reason to accuse an author of "serene omninescience of the progress made since the middle of the eighteenth century."

In the second place the authors seem to be oblivious of the fact that mathematics intends to be an abstract science and that it is not primarily concerned with the concrete instances from which the abstractions with which it deals may be made. The larger part of the present volume, stripped of the parts irrelevant to its own purpose, is occupied with finding concrete instances for the abstractions number, variable, and function, upon the basis of vector analysis. The classification of quantities into sorts, kinds, and varieties is without significance for the numbers of mathematics, though perhaps useful if one wishes to provide a concrete basis for each of the types of numbers which occur. To do this may be of interest to the pedagogue, but it is without importance for the mathematician and of questionable value for the scientific philosopher, particularly when for the genesis of the vector analysis itself at least the real number system would be requisite. It is not surprising that the authors can not be satisfied with the definitions of variable and function which they find in mathematical books, when we realize that the writers of these books are not dealing with the things their critics seem to be interested in.

And in the third place it appears to the reviewer that the method pursued by this book is entirely inadequate for its purpose. It is recognized generally that in a systematic discussion of the foundations of any division of mathematics, we must begin by knowing precisely what it is we are going to talk about and what our fundamental assumptions are going to be, i. e., we begin by laying down indefinables and postulates. This the authors of this book have not done and the result is that in many places we are at a loss to determine the meaning of the words used, and that, if we make a correct guess at their meaning, we find ourselves confronted with numerous instances of "vicious circle definitions," i. e., definitions which do not define. We call attention to the discussion of sorts on pages 24 and 25 (what does it mean to add two quantities?); to that of kinds on page 27 (when is one quantity equal to, greater than or less than another?); to the discussion of units on page 39 (what is $+1$ when we do not yet know what a unit is?) and to the definition (?) of sort on page 76 et seq., as well as to the discussion of the meaning of zero on page 139.

That the authors do not have any respect for authority will perhaps be regarded as a merit by many, although the manner in which this "originality" finds utterance can not fail to irritate the mathematical reader, particularly when positive evidence of a lack of mathematical maturity comes before him. Such evidence is found in the meaningless discussion on page 188 which aims to "make plain to the veriest tyro in mathematics" that the definition of function as given, for instance, in Osgood's Funktionentheorie is "erroneous"; also in the discussion of Riemann's definition of a function of a complex variable on pages 196 and 197.

The reviewer considers the appearance of this book therefore as a distinctly unfortunate occurrence and he hopes that wiser counsel may prevail if the publication of future parts of the work should be under consideration. Poincaré, Russell, and others have done much to bridge the gulf which has separated philosophy and mathematics. Their work has undoubtedly contributed a great deal on the one hand to the appreciation on the part of mathematicians of the problems in the foundations of their science, and on the other hand to the initiation of a new tendency in philosophy. Further work in this field will therefore be received with interest and hope by mathematicians, provided it give clear evidence of the author's competence in both mathematics and philosophy. Unless this be the case, the workers in the two fields will be driven apart, for each group is inclined unfortunately to hold the other group responsible for the acts of any one whom they have reason to suspect of belonging to that group. It is earnestly to be hoped that the mathematicians who may happen to read the volume under review may not base upon it a general condemnation of the utterances of philosophers concerning mathematics, but will give themselves an antidote in the form of such books as Russell's Scientific Method in Philosophy or Holt's Concept of Consciousness.

ARNOLD DRESDEN.

Homogeneous Linear Substitutions. By HAROLD HILTON, M.A., D.Sc. Oxford at the Clarendon Press. 1914. Pp. 184.

PROFESSOR Hilton's book is a welcome addition to the text-book literature on the subject of linear substitutions. In the preface the author states that he has "attempted to put together for the benefit of the mathematical student those properties of the homogeneous linear substitution with real or complex coefficients of which frequent use is made in the theory of groups and in the theory of bilinear forms and invariant factors."

The first four chapters, comprising a little more than half of the book, are intended to form an introduction to the whole subject. In the first chapter, which is much the longest in the book, the ordinary method of transforming the general substitution into the normal and canonical forms by means of the poles is shown and the simpler properties of symmetric, orthogonal, unitary, and Hermitian substitutions are given. In the second the author gives a very brief account of invariant factors* and develops the second canonical form which is the direct product of substitutions of the type

$$x_1' = x_2, \quad x_2' = x_3, \cdots, x_r'{}_{-1} = x_r,$$

$$x_r' = e_1x_1 + e_2x_2 + \cdots + e_rx_r.$$

In the third chapter devoted to bilinear forms the Hermitian forms play a prominent part.

To the student who comes to the subject for the first time the fourth chapter on Applications will be one of the most interesting in the book. Illustrations from the theory of equations, from differential equations, from the theory of maxima and minima, from geometry, and from mechanics serve to show the wide range of application of the subject.

* Following Bromwich, Hilton uses the term "invariant factor" instead of "elementary divisor."