THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY AT CHICAGO.

The thirty-ninth regular meeting of the Chicago Section, the eighth meeting of the Society at Chicago, was held at the University of Chicago on Friday and Saturday, April 6 and 7. About seventy-five persons were present, among them the following fifty-five members of the Society:

Professor G. A. Bliss, Professor R. L. Borger, Professor P. P. Boyd, Professor J. W. Bradshaw, Professor W. H. Bussey, Professor W. D. Cairns, Professor R. D. Carmichael, Dr. E. W. Chittenden, Mr. E. H. Clarke, Dr. L. C. Cox, Professor A. R. Crathorne, Mr. G. H. Cresse, Professor D. R. Curtiss, Professor L. E. Dickson, Professor A. Dresden, Professor W. B. Ford, Dr. W. V. N. Garretson, Dr. G. H. Graves, Dr. J. O. Hassler, Mr. C. M. Hebbert, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Mr. Glenn James, Professor A. M. Kenyon, Professor W. C. Krathwohl, Professor Kurt Laves, Professor W. D. MacMillan, Professor W. Marshall, Professor T. E. Mason, Dr. A. L. Miller, Professor G. A. Miller, Professor E. H. Moore, Professor E. J. Moulton, Professor F. R. Moulton, Dr. J. R. Musselman, Dr. A. L. Nelson, Dr. F. W. Reed, Professor H. L. Rietz, Professor W. H. Roever, Miss I. M. Schottenfels, Dr. A. R. Schweitzer, Professor J. B. Shaw, Professor H. E. Slaught, Mr. G. W. Smith, Professor E. J. Townsend, Professor A. L. Underhill, Professor E. B. Van Vleck, Dr. V. H. Wells, Professor E. J. Wilczynski, Dr. C. E. Wilder, Professor R. E. Wilson, Professor C. H. Yeaton, Professor A. E. Young, Professor J. W. A. Young, Professor A. Ziwet.

The sessions on Friday were presided over by Professor W. B. Ford, chairman of the Section; the session of Saturday forenoon was under the chairmanship of Professor L. E. Dickson, President of the Society. At the dinner held at the Quadrangle Club on Friday evening, fifty-one persons were present.

Friday afternoon was devoted to a symposium on "The Lebesgue Integral." The idea of a symposium was first suggested by Professor Van Vleck in April, 1915; it had been under discussion by different committees of the Chicago Section since that time. The committee appointed in April, 1916, for the purpose of arranging a symposium for the present meeting consisted of Professors Van Vleck and C. N. Moore, together with the members of the program committee.

The principal papers were presented by Professor Bliss and Professor Hildebrandt. Since these papers are to appear in full in later numbers of the BULLETIN, only a list of the topics treated by them will be given here.

Professor Bliss, "Integrals of Lebesgue and their applications": 1. Definition and existence of a Lebesgue integral; 2. Measure of point sets and measurable functions; 3. Properties of Lebesgue integrals; 4. Some applications of Lebesgue integrals.

Professor Hildebrandt, "Integrals, extensions of and related to the integrals of Lebesgue": 1. Integrals defined on non-measurable sets; 2. Extension of integration to non-summable functions; 3. The Stieltjes integral; 4. The Hellinger integral; 5. Generalizations of Lebesgue, Stieltjes, and Hellinger integrals.

The papers were discussed by Professor E. H. Moore and Professor E. R. Hedrick.

The forenoons of Friday and Saturday were devoted to the presentation of the following papers:

(1) Professor T. H. HILDEBRANDT: "On boundary value problems for linear differential equations in general analysis."

(2) Dr. E. W. CHITTENDEN: "On the relation of non-metrical analysis situs to the calcul fonctionnel of Fréchet."

(3) Dr. A. L. MILLER: "Projective differential geometry of three dimensional varieties in S_5 and of line complexes."

(4) Professor R. D. CARMICHAEL: "Comparison theorems for homogeneous linear differential equations of general order."

(5) Professor R. D. CARMICHAEL: "Note on convergence tests applicable to series converging conditionally."

(6) Mr. W. H. WILSON: "On a certain general class of functional equations."

(7) Dr. A. R. SCHWEITZER: "Functional equations based on iterative compositions" (second paper).

(8) Dr. A. R. SCHWEITZER: "On the implicit correlatives of certain functional equations" (preliminary communication).

(9) Mr. C. M. HEBBERT: "Circular curves generated by pencils of stelloids and their polars."

(10) Professor R. D. CARMICHAEL: "On the representation of functions in series of the form $\Sigma c_n g(x + n)/g(x)$."

(11) Dr. A. J. KEMPNER: "A theorem on lattice points."

(12) Dr. A. J. KEMPNER: "On irreducible equations admitting roots of the form $a + re^{i\theta}$, a rational and r rational."

(13) Mr. G. W. SMITH: "Nilpotent algebras generated by two generators, i and j, such that i^2 is not an independent unit."

(14) Professor J. B. SHAW: "Simplification of defining equations of nilpotent algebras."

(15) Professor E. B. STOUFFER: "On the calculation of invariants and covariants of linear homogeneous differential equations."

(16) Professor G. A. MILLER: "Substitution groups and possible arrangements of players at card tournaments."

(17) Professor T. E. MASON: "On functional solutions of certain diophantine equations" (preliminary communication).

(18) Dr. A. J. KEMPNER: "On the zeros of integral transcendental functions belonging to certain simple classes and the zeros of the derived functions."

(19) Mr. J. L. WALSH: "On the solution of linear equations in infinitely many variables by successive approximations."

(20) Dr. A. L. NELSON: "Plane nets and conjugate systems of curves."

(21) Dr. V. H. WELLS: "Single parameter systems of polar fields."

(22) Professor S. LEFSCHETZ: "On certain two-dimensional cycles belonging to an algebraic surface."

Mr. Wilson was introduced by Professor Carmichael, and Mr. Walsh by Professor Van Vleck. The papers of Professor Stouffer, Dr. Kempner, and Professor Lefschetz were read by title.

Abstracts of the papers, numbered to correspond to the titles in the list, follow below.

1. In this paper Professor Hildebrandt considers linear boundary conditions relative to the general linear differential equation of the form

$$D\eta = \alpha_0 - J\alpha\eta$$

in which η, α_0, and α are functions belonging to a class $\mathfrak{H}$, of the three variables p', p'' and x, which range over the classes $\mathfrak{P}'$ and $\mathfrak{P}''$ (general classes) and $\mathfrak{X}$ (a linear interval) respectively; $D = d/dx$, and J is a linear operator on a class $\mathfrak{K}$ of functions on the range $\mathfrak{P}''\mathfrak{P}'$. The boundary conditions considered are of the form

$$c\eta(x_1) + J\sigma_1\eta(x_1) + d\eta(x_2) + J\sigma_2\eta(x_2) = \sigma_0$$

in which c and d are constants satisfying the condition $c+d=1$, σ_1, σ_2 and σ_0 are functions of the class $\mathfrak{K}$, and x_1 and x_2 are any two elements of $\mathfrak{X}$. The conditions under which there exist solutions of the differential equation and the boundary conditions are determined. By a suitable definition of adjoint boundary conditions relative to the adjoint differential equation

$$D\hat{\eta} = \alpha_0 + J\hat{\eta}\alpha$$

the usual theorems concerning the interrelations of the solutions of adjoint systems are obtained.

2. Dr. Chittenden shows for a class S, satisfying any of the sets of axioms Σ_1, Σ_2, Σ_3 given by Professor R. L. Moore as foundations for a theory of plane curves in non-metrical analysis situs, that it is possible to define the écart (P, Q) of any two elements of S, so that if $P = L_n P_n$ then $L_n(P_n, P) = 0$, and conversely. There will exist classes S_0, S_1, S_2, relative to each element P of S, such that S_0 contains P; S_2 contains S_0, S_1, and no limiting element of S_0. Hahn has shown that there exists a function $f(Q)$ which vanishes on S_0, is identically 1 on S_2, and satisfies the inequality $0 < f(Q) < 1$ on S_1. It is shown in the present paper that the solutions of the equation $f(Q) = x$, $0 < x < 1$, form in the aggregate a simple closed curve whose interior contains S_0, and therefore P.

These considerations show that non-metrical analysis situs as studied by Veblen, Lennes, and R. L. Moore is included in the calcul fonctionnel of Fréchet. A theory equivalent to the theories of these writers can therefore be based on the undefined notions point and distance. The author is continuing his investigations in this direction.

3. In this paper Dr. Miller sets up the following completely integrable set of four homogeneous linear partial differential equations in three independent variables:

$$\frac{\partial^2 y}{\partial u_1^2} = a_0^{11} y + a_1^{11} \frac{\partial y}{\partial u_1} + a_2^{11} \frac{\partial y}{\partial u_2} + a_3^{11} \frac{\partial y}{\partial u_3},$$

$$\frac{\partial^2 y}{\partial u_1 \partial u_2} = a_0^{12} y + a_1^{12} \frac{\partial y}{\partial u_1} + a_2^{12} \frac{\partial y}{\partial u_2} + a_3^{12} \frac{\partial y}{\partial u_3} + a_4^{12} \frac{\partial^2 y}{\partial u_1 \partial u_3} + a_5^{12} \frac{\partial^2 y}{\partial u_2 \partial u_3},$$

$$\frac{\partial^2 y}{\partial u_2^2} = a_0^{22} y + a_1^{22} \frac{\partial y}{\partial u_1} + a_2^{22} \frac{\partial y}{\partial u_2} + a_3^{22} \frac{\partial y}{\partial u_3},$$

$$\frac{\partial^2 y}{\partial u_3^2} = a_0^{33} y + a_1^{33} \frac{\partial y}{\partial u_1} + a_2^{33} \frac{\partial y}{\partial u_2} + a_3^{33} \frac{\partial y}{\partial u_3},$$

where the a_k^{ij}'s satisfy certain integrability conditions. The six independent solutions of these equations can be looked upon as the homogeneous coordinates of a point in five dimensions, depending on three parameters, and therefore represent a three-dimensional variety V_3 in a five-dimensional space S_5. The projective differential properties of this V_3 may be studied by means of the invariants and covariants of this set of equations under the transformations

$$\bar{y} = \lambda y \quad \text{and} \quad \bar{u}_i = \overline{U}_i(u_i).$$

A fundamental set of these invariants and covariants has been found, as well as operators that produce from this set all other invariants and covariants.

If the V_3 lies on a hyperquadric in S_5 there is a one-to-one correspondence between the points of it and the lines of a complex in S_3. Thus the differential properties of a line complex invariant under the general linear transformation on Klein coordinates may also be studied by means of this set of equations.

The only cases ruled out are those in which the V_3 lies in an S_4 or is a developable of the second kind.

4. This paper by Professor Carmichael deals with several theorems of comparison stating interesting relations between the solutions of two homogeneous linear differential equations of general order such that the coefficients of the one equation bear certain prescribed relations to those of the other equation. Among the results obtained are extensions in several directions of one of Sturm's classical theorems of comparison

for equations of the second order. (See Sturm, *Journal de Mathématiques*, 1 (1836): 106–186, and Bôcher, BULLETIN, 4 (1898): 295–313, 365–376.)

5. In this paper Professor Carmichael derives a general theorem concerning the convergence of series which need not converge absolutely and applies it to obtain some more usable particular theorems of which the following is one: If the series $a_1 + a_2 + a_3 + \cdots$ is summable of order $k - 1$ in the sense of Cesàro and if constants $c_1, c_2, c_3, \cdots$ are selected so that the series

$$\sum_{i=1}^{\infty} i^{k-1} \mid \Delta^k c_i \mid$$

converges, the limit

$$\lim_{n=\infty} n^{k-1} \Delta^{k-1} c_{n+1}$$

exists and is finite, and the quantities

$$n^{k-1} \Delta^{k-\tau} c_{n+1}, \quad (\tau = 2, 3, \cdots, k)$$

are bounded, then the series

$$a_1 c_1 + a_2 c_2 + a_3 c_3 + \cdots$$

converges (but does not necessarily converge absolutely).

6. The principal results of Mr. Wilson's paper may be stated as follows: Every solution $f(x)$ of the equation

$$(1) \qquad \sum_{i=1}^{n} \gamma_i f(\alpha_i x + \beta_i y) + \gamma_{n+1} f(x) + \gamma_{n+2} f(y) = 0,$$

in which the α's, β's and γ's are constants and no α is zero, is a solution of the normal equation

$$(2) \qquad \sum_{k=0}^{n+1} (-1)^{n+1-k} \frac{(n+1)!}{k!(n+1-k)!} f(kx + y) = 0.$$

The most general solution of (2) continuous over the finite complex x-plane is a polynomial in u and v of degree n with arbitrary coefficients, where $x = u + \sqrt{-1}\, v$, and u and v are real. The general solution of (2) analytic over the finite complex x-plane is a polynomial in x of degree n with arbitrary coefficients. The most general solution of (2) continuous along any line not parallel to the axis of imaginaries is a polynomial in u of degree n with arbitrary coefficients, while the most general solution continuous along any line not parallel to the axis of reals is a polynomial in v of degree n with arbitrary coefficients. From these properties of the normal equation it is easy to derive the corresponding properties of equation (1). Besides these several other results are obtained incidentally.

7. Let $\{F[0, 1, 2, \cdots, n; i_1^{(k)} i_2^{(k)} \cdots]\}$ $(k = 1, 2, \cdots)$ be a set of iterative compositions of a function of $n + 1$ variables $(n = 1, 2, 3, \cdots)$, the variables in the entire set being notationally distinct. Then by an equation in iterative compositions Dr. Schweitzer means a relation between the preceding compositions F and some, none, or all of the variables explicitly involved. By means of suitable adjoined conditions on the variables involved special types of equations are obtained, e. g., the "equation in partial functions" of Herschel.* Generalization of the preceding to simultaneous systems is made.

An important class of equations in iterative compositions are those which involve infinite sequences and consequently the notion of convergence.† In particular, limiting cases of certain generalizations of functional equations previously defined by the author are obtained. Example: If

$$f_t^{(n)}(x) = f[t, f\{t, \cdots, f(t, x)\}],$$

where in the latter iteration the f occurs n times, and

$$\underset{n \quad \infty}{L} f[f\{f_t^{(n)}(x), x\}, f\{f_t^{(n)}(y), y\}] = f(y, x),$$

then a particular solution is

$$f(x, y) = \psi^{-1}\left\{\frac{\psi(x) - \psi(y)}{2}\right\}.$$

Finally, classes of eliminative functional equations are constructed which are satisfied by the solution of the equation $f_{t_1 t_2 \cdots t_n}{}^{(k)}(x) = x$, where $k = 2, 3, 4, \cdots$, and $n = 1, 2, 3, \cdots$.

* Cf. *Philosophical Transactions*, 1814, pp. 458 ff.

† Cf. the memoirs of Nicoletti, especially, *Mem. mat. fis. Soc. Ital. delle Scienze* (3), 14 (1906). The subject matter of this memoir is capable of further abstraction.

Here for $n = 2$, $k = 2$, $f_{t_1t_2}^{(2)}(x) = f\{t_1, t_2, f(t_1, t_2, x)\}$, etc. Such a class is, for instance, $f\{f_t^{(k)}(x), f_t^{(k)}(y)\} = f(x, y)$, which has the particular solution $f(x, y) = \psi^{-1}\{\xi\psi(x) + \xi\psi(y)\}$, where $\xi^{k-1} + \xi^{k-2} + \cdots + \xi^2 + \xi + 1 = 0$.

8. An important aspect of iterative compositions and functional equations is their possible expression in terms of transformations of variables and properties of these transformations, e. g., iteration of transformations. Dr. Schweitzer makes a number of applications of this theory based on equations previously defined by him. One application consists in expressing certain of his eliminative functional equations in implicit form and interpreting the solution of the latter as a problem of the inversion of the process of elimination of variables in algebra and analysis. Stated in somewhat general form, the problem is as follows: Given that an eliminant of the relations $f_i\{x_1', x_2', \cdots, x_i', x_1, x_2, \cdots, x_i, t_1, t_2, \cdots, t_k\} = 0$ $(k < i;\ i = 2, 3, \cdots)$ is $\psi\{x_1', x_2', \cdots, x_i', x_1, x_2, \cdots, x_i, f_1, f_2, \cdots, f_i, \xi\} = 0$, to find the functions f_i. Example: To determine ϕ in the equations $\phi(x_1', x_1, t) = 0$, $\phi(x_2', x_2, t) = 0$, $\phi(\xi, x_1', x_2') - \phi(\xi, x_1, x_2) = 0$, which is an implicit form of the equations $x_1' = f\{x_1, t\}$, $x_2' = f\{x_2, t\}$, $f\{x_1', x_2'\} = f\{x_1, x_2\}$. Another application consists in the solution of quasi-transitive functional equations as defined by transformations of types occurring in the theory of Lie.*

9. Mr. Hebbert studies the circular curves generated by pencils of stelloids which have the $(n + 1)$st roots of unity and their associates as base points and are connected with the transformation $z' = 1/z^n$. This is a special case of the transformation $z' = z - (n + 1)f(z)/f'(z)$, discussed by Professor Emch in the *Rendiconti del Circolo Matematico di Palermo*, volume 34 (1912), pages 1–12.

10. In this paper Professor Carmichael considers the general problem of representing functions with certain assigned properties at infinity in series of the form

$$\sum_{n=0}^{\infty} c_n \frac{g(x+n)}{g(x)},$$

* Cf., e. g., Levi-Civita, *Real. Inst. Lomb. Rendiconti* (2), vol. 28 (1895); Vessiot, same volume.

where $g(x)$ is a given function having the asymptotic representation

$$g(x) \sim x^{\mu - x} e^{\alpha + \beta x} \left(1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots \right),$$

valid in a sector V including the positive axis of reals in its interior and being analytic in V for sufficiently large values of $|x|$. This problem contains as a special case the problem of the representation of functions in factorial series. (See *Transactions,* volume 17 (1916), pages 207–232.)

11. In this paper, Dr. Kempner discusses the system of rectilinear paths of finite width and extending to infinity in both directions which may be laid through a square lattice-point system.

Assuming in a rectangular system of coordinates the lattice-points represented by $x = 0, \pm 1, \pm 2, \cdots, y = 0, \pm 1, \pm 2, \cdots$, and considering the width d of the path as a function of the slope $\tan \varphi$, the following theorem is easily derived: When $\tan \varphi = p/q$, p, q relatively prime, then the broadest path in the direction φ which does not contain any points in its interior has the width $d = (p^2 + q^2)^{-\frac{1}{2}}$. When $\tan \varphi$ is irrational, no path of finite width exists which does not contain an infinite number of points in its interior.

Writing $d = f(\tan \varphi)$, or $y = f(x)$, it is seen that $f(x)$ is a pointwise discontinuous nullfunction which is closely related to the well known pointwise discontinuous function $y = F(x)$, $F(x) = 0$ for x irrational, $F(p/q) = 1/q$ for p, q relatively prime.

12. The present paper is a continuation of a paper: "On irreducible equations," presented by Dr. Kempner to the Southwestern Section in November, 1913. It deals with irreducible equations having among their roots some of the form $a + re^{i\theta}$, a and r both rational. Irreducibility here refers to the natural domain although extension to other domains is immediate for most of the theorems. Consider all circles in the plane, of rational radii about all real rational points as centers. One obtains thus a triply infinite system of circles which cover everywhere densely the complex plane. Some of the theorems established may then be expressed as follows: When an irreducible equation has roots on any of the circles, then

I. The equation must be of even degree;

II. There are just as many roots lying outside the circle as there are inside, and the roots outside are the points of reflection of the points inside.

III. The equation cannot have roots on any other circle which does not intersect the first; but

IV. The equation may have roots on another circle which intersects the first;

V. The radius and the center of the circle may always be determined by a finite number of rational operations and extraction of radicals;

VI. The equation is completely solvable by radicals when its degree is smaller than ten;

VII. Comparatively simple necessary and sufficient conditions for the existence of roots on any of the circles are established.

13. In this paper Mr. Smith determines the types of nilpotent algebras which are generated by two units, i and j, where i^2 is not an independent unit. He expresses these units in terms of the associative units of Professor Shaw and then forms the products ji and i^2. Since in the form

$$i,\ j,\ ij,\ j^2,\ ij^2,\ \cdots,\ j^{\mu_2-1},\ ij^{\mu_2-1},\ j^{\mu_2},\ j^{\mu_2+1},\ \cdots,\ j^{\mu_1-1}$$

the product of any two units is linearly expressible in terms of the units which follow both factors, certain relations among the parameters arise from the products ji and i^2. The solution of these relations leads to thirty different types of nilpotent algebras. Certain general theorems permit a simplification in the expressions for the units and the classification becomes possible.

14. In this paper Professor Shaw shows that the defining equations of a Peirce algebra (nilpotent system of one partial modulus) admit of simplifications due to a choice of units which is usually possible. The advantage of the simplification lies in the greater ease of handling the representation in associative units. If the equations in the regularized base units and the adjunct unit j are

$$ji_a = \sum_{b=1}^{m} i_b j P_{ba} + j^2 P_a \quad (a = 1, \cdots, m),$$

where P_{ba} and P_a are polynomials in j, then either the determinant

$$\begin{vmatrix} P_{11}-1, & P_{12} \quad , & \cdots \\ P_{21} \quad , & P_{22}-1, & \cdots \\ \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \end{vmatrix}$$

vanishes or does not vanish. In the latter case the base units i may be so chosen as to reduce P_{ba} to j merely for $b = 1, \cdots, m$. In the former case when the second term in the principal diagonal elements is replaced by λ, the determinant set equal to 0 has m roots, which are polynomials in j. If these are all distinct the terms in j^2P_a all drop out for a properly chosen base. If the roots are not distinct the problem needs further study.

15. In previous papers presented to the Society, Professor Stouffer has obtained complete systems of seminvariants and semicovariants for a system of ordinary linear homogeneous differential equations of the second order with n dependent variables. In the present paper the problem is completed by the calculation of complete systems of invariants and covariants for this system of differential equations. The more complicated invariants and covariants are obtained by applying certain operators to a set of simple invariants and covariants.

16. To emphasize certain properties of substitution groups Professor Miller directs attention to their availability for the purpose of determining arrangements of the players at card tournaments so that during a series of games each player will have each of the others once as a partner and twice as an opponent. The possibility of determining such arrangements was considered, from a different point of view, by E. H. Moore in his "Tactical memoranda" published in volume 18 of the *American Journal of Mathematics*.

When the number of players is a power of 2, sets of possible arrangements satisfying the stated condition can be obtained directly by means of the regular substitution group of order 2^m, and of type $(1, 1, 1, \cdots)$, by starting with any subgroup of order 4 contained in such a substitution group and transforming this subgroup under the powers of an operator of order $2^m - 1$ in the group of isomorphisms of this substitution group. When m is even we may impose the additional condition that during the first $(2^m - 1)/3$ games each of the players has each of the others once and only once either as a partner or as an opponent, but sets in which this additional condition is not satisfied can be obtained with equal ease in this case.

17. In this paper Professor Mason replaces, in certain equations of Diophantus, the unknowns which are to be rational numbers by unknown functions restricted only by the equations. The functional equations are then solved. These functional equations, as such, are of some interest. The problem remaining is to restrict these functional solutions so that they will satisfy the conditions of Diophantus. The number of diophantine equations to which this process may be applied seems large.

18. In the first part of this paper Dr. Kempner unifies a set of known theorems (comprising, for example, the first two theorems of § 230, and the theorems of §§ 232, 233, 234 in Vivanti-Gutzmer, Analytische Funktionen, 1906); their proofs are slightly simplified by introducing convex polygons (open or closed) which extend to infinity in the complex plane. As an application, the following theorem is derived:

Let $f(z) = c_0 + c_1z + c_2z^2 + \cdots$, $c_0 \neq 0$, converge in the whole complex plane, let a_ν, $(\nu = 1, 2, \cdots)$, be the zeros of $f(z)$, and let $\Sigma |a_\nu|^{-\epsilon}$ be convergent for every $\epsilon > 0$; if any one of the $c_1, c_2, \cdots$ is zero, then either the zeros of $f(z)$ lie all on a straight line passing through the origin, or every straight line through the origin has zeros on either side. (All a and c are real or complex.) In the second part theorems of the following kind are derived: Let m, n be any two positive integers, and $a_1, \cdots, a_n$ any n numbers, $b_1, \cdots, b_m$ any m numbers, real or complex, and arbitrary except that no two a's shall be equal, and no a equal to any b. Then a function $f(z)$ exists such that

I. $f(z) = e^{g(z)} \cdot \varphi(z)$, where $g(z)$ and $\varphi(z)$ are polynomials;

II. $f(z) = 0$ has exactly the roots $a_1, \cdots, a_n$;

III. $f'(z) = 0$ has exactly the following roots: (a) $b_1, \cdots, b_m$ (all single, or counted with proper multiplicity when $b_1, \cdots, b_m$ are not all distinct), (b) up to $n - 1$ other roots, which cannot be arbitrarily assigned and which in general depend both on the given a and b.

The author does not believe that this theorem can be derived by specialization from a (possibly related) theorem of much wider scope due to Guichard (*Annales de l'Ecole Normale Supérieure,* 1884).

19. Mr. Walsh considers a system of equations

$$x_i + \sum_{j=1}^{\infty} a_{ij} x_j = c_i \quad (i = 1, 2, 3, \cdots)$$

under the restrictions

$$\sum_{j=1}^{\infty} |a_{ij}| \leqq p < 1, \quad |c_i| \leqq C.$$

A solution is obtained by using successive approximations; this solution is such that the x's are bounded, and it is the only such solution.

By a slight change in the form of the equations, the results are extended to include some well known results of Poincaré and von Koch.

20. G. Koenigs has proved that the perspectives on a fixed plane from a fixed point of the asymptotic curves of a surface form a net with equal Laplace-Darboux invariants. Dr. Nelson's paper deals with the perspectives of conjugate systems of curves on a surface. Use is made of Wilczynski's methods, according to which the conjugate system of curves is considered as being generated by the point P_y whose homogeneous coordinates, $y^{(k)} = y^{(k)}(u, v)$ $(k = 1, 2, 3, 4)$, constitute a set of linearly independent solutions of the completely integrable system of equations

$$y_{uu} = a y_{vv} + b y_u + c y_v + d y,$$
$$y_{uv} = b' y_u + c' y_v + d' y.$$

A point, P_τ, which describes a plane net of curves, has the homogeneous coordinates $\tau^{(k)} = \tau^{(k)}(u, v)$ $(k = 1, 2, 3)$, which are linearly independent solutions of the completely integrable system

$$(T) \qquad \begin{aligned} \tau_{uu} &= A\tau_u + B\tau_v + C\tau, \\ \tau_{uv} &= A'\tau_u + B'\tau_v + C'\tau, \\ \tau_{vv} &= A''\tau_u + B''\tau_v + C''\tau. \end{aligned}$$

The following theorems are obtained. (*a*) Any plane net may be considered as the perspective from a fixed point of a conjugate system of surface curves. (*b*) The perspective on a fixed plane from a fixed point of a given conjugate system of surface curves forms a plane net whose completely integrable system (T) is characterized by the mixed invariantive relation

$$\frac{\partial^2}{\partial u \partial v} \log a - \frac{\partial}{\partial v}(aA'') + \frac{\partial}{\partial u}\left(\frac{B}{a}\right) - \frac{\partial A'}{\partial u} + \frac{\partial B}{\partial v} = 0.$$

21. The paper of Dr. Wells is a study of the five single parameter systems of polar fields, the pencil, the range, and the three intervening mixed systems. The pencil of polar fields is defined as all the polar fields determined by triangles perspective and reciprocal to one triangle, such that the corresponding sides of those triangles form three projective first order pencils of rays. Similar definitions are given for the other forms by means of the determining triangles. In the pencil of polar fields the centers of perspective for the determining triangles form a range of the fourth order, and the lines of perspective a pencil of the second order. For each system, the principal theorems give the loci of the poles of a line, and the polars of a point. In the first mixed system the polars of a point form a second order pencil, and the poles of a line a fourth order range.

22. In this note Professor Lefschetz establishes the existence of a two dimensional cycle Γ, corresponding to any two algebraic curves C_1, C_2 on an algebraic surface $F(x, y, z) = 0$. By means of certain functions of Poincaré, the periods of a double integral

$$(1) \qquad \int\int \frac{P(x, y, z)}{F_z'} dxdy \quad (P \text{ adjoint polynomial})$$

are found. From these it follows that if $t_1 C_1 = t_2 C_2$, $\Gamma \sim 0$. If (1) is of the form

$$\int\int \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right) dxdy,$$

the periods are linear combinations of the logarithmic periods of

$$\int V(x, \bar{y}, z) dx; \quad F(x, \bar{y}, z) = 0,$$

from which a very interesting connection with Severi's theory of the base is derived. The paper appeared in the *Rendiconti dei Lincei* for February.

ARNOLD DRESDEN,
Secretary of the Section.

THE APRIL MEETING OF THE SAN FRANCISCO SECTION.

THE twenty-ninth regular meeting of the San Francisco Section was held at Stanford University on Saturday, April 7. Two sessions were required for the presentation of the program. Professor Lehmer occupied the chair. The following members of the Society were present:

Professor R. E. Allardice, Professor H. F. Blichfeldt, Dr. Thomas Buck, Professor M. W. Haskell, Professor L. M. Hoskins, Professor D. N. Lehmer, Professor W. A. Manning, Professor H. C. Moreno, Professor C. A. Noble, Professor E. W. Ponzer, Dr. H. N. Wright.

It was voted to hold the next meeting of the Section at the University of California, on October 27.

The following papers were read at this meeting:

(1) Professor H. F. BLICHFELDT: "A further reduction of the known maximum limit to the least value of quadratic forms."

(2) Professor D. N. LEHMER: "Certain divisibility theorems concerning the convergents of Hurwitzian continued fractions."

(3) Dr. G. F. MCEWEN: "Determination of the functional relation between one variable and each of a number of correlated variables by successive approximation."

(4) Professor W. A. MANNING: "On the order of primitive groups (III)."

(5) Dr. H. N. WRIGHT: "Note on a certain quadratic transformation of the plane."

In the absence of Dr. McEwen, his paper was read by Mr. E. F. Michael of the Scripps Institute for Biological Research. Abstracts of the papers follow below.

1. Having given the determinant D of a positive-definite quadratic form F in n variables, such integers, not all zero,