THE PROJECTION OF A LINE SECTION UPON THE RATIONAL PLANE CUBIC CURVE.

BY PROFESSOR JOSEPH EUGENE ROWE.

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Introduction.

THE rational plane curve of the third order, which we shall refer to as the R^3, is of the fourth class; that is, from an arbitrary point of the plane four tangents can be drawn to the curve. But if the point is selected on the R^3 itself, the tangent at the point accounts for two of these tangents, and, therefore, from such a point only two additional tangents can be drawn to the curve. A line section yields three points of the R^3 and these, in the manner just described, determine three pairs of additional tangents. An investigation of the points of a line and the six tangents so determined shows that the relations which exist among these are interesting as well as of a fundamental character.

We shall let

$$x_i = a_it^3 + 3b_it^2 + 3c_it + d_i \qquad (i = 0, 1, 2) \tag{1}$$

be the parametric equations of the points of the R^3, and it has been found convenient to use the following abbreviations:

$$\alpha = | abc |, \quad \beta = | abd |, \quad \beta' = | acd |, \quad \alpha' = | bcd |. \tag{2}$$

Also, it may be verified that the identities

$$a_i\alpha' - b_i\beta' + c_i\beta - d_i\alpha = 0 \tag{3}$$

exist among the coefficients in (1) and the Greek letters of (2).

The Choice of a Line Section.

As the parameters 0 and ∞ may be assigned to any two elements in a one-dimensional space, we select the line determined by the points of the R^3 whose parameters are 0 and ∞. From (1) it follows that the coordinates of these points are d_i and a_i, respectively; hence the equation of the line determined by them is $| adx | = 0$, and the parameter of the third point of the R^3 (found by substituting from equations (1) in $|\, adx \,| = 0$) collinear with a_i and d_i is the root of

$$(4) \qquad \beta t + \beta' = 0.$$

By substituting $t = -\beta/\beta'$ in (1) we obtain for the coordinates of the point (4)

$$(5) \qquad x_i = -a_i\beta'^3 + 3b_i\beta'^2\beta - 3c_i\beta'\beta^2 + d_i\beta^3 \quad (i=0, 1, 2).$$

The Projections of the Three Points upon the R^3.

The projection of a point x_i upon (1) is*

$$(6) \qquad |\, abx \,|\, t^4 + 2\,|\, acx \,|\, t^3 + (|\, adx \,| + 3\,|\, bcx \,|)t^2 + 2\,|\, bdx \,|\, t + |\, cdx \,| = 0.$$

That is, if the coordinates of a point x_i are substituted in (6), the result is a quartic in t whose roots are the parameters of the points of contact of the four tangents that can be drawn from x_i to the R^3 of (1).

By substituting a_i, d_i, and the coordinates (5) in (6) for x_i we obtain

$$(7) \qquad 3\alpha t^2 + 2\beta t + \beta' = 0,$$

$$(8) \qquad \beta t^2 + 2\beta' t + 3\alpha' = 0,$$

$$(9) \qquad (\beta^2 - 3\alpha\beta')t^2 + (3\alpha'\beta - \beta'^2) = 0,$$

whose roots are the parameters of the points of contact of the tangents to R^3 drawn from the points whose parameters are ∞, 0, and $-\beta'/\beta$, respectively.

The form of equations (7)–(9), if properly interpreted, conveys a great deal of information. Evidently the roots of (9)† are harmonic‡ (apolar) to 0 and ∞; the roots of (8) are harmonic to ∞ and $-\beta'/\beta$; and the roots of (7) are harmonic to 0 and $-\beta'/\beta$. These results we summarize in

THEOREM I. *The parameters of any two of three collinear points on the R^3 are harmonic to the parameters of the points of contact of the two additional tangents that can be drawn to the R^3 from the third point.*

* J. E. Rowe, BULLETIN, vol. 22, No. 2, p. 75 (November, 1915).

† Observe that the order of statement in this sentence is not without purpose.

‡ Salmon, Higher Algebra, fourth edition, p. 179.

Also, the determinant of equations (7)–(9) vanishes, as may be seen at once from the fact that (9) may be obtained by subtracting β' times (7) from β times (8). Hence* we have

THEOREM II. *The parameters of the points of contact of the three pairs of tangents that can be drawn to the R^3 from three collinear points of the R^3 are harmonic to the same quadratic, or form a set in involution.*

Another result which may be derived as a corollary of Theorem I we shall state as

THEOREM III. *Lines on a point P of an R^3 cut the R^3 in pairs of residual points whose parameters are harmonic to the parameters of the points of contact of the two additional tangents drawn to R^3 from P.*

Although Theorem III may be regarded a corollary of Theorem I, it may be established independently. Thus: Let $P(d_0, d_1, d_2)$ be the point and $(\kappa x) \equiv \kappa_0 x_0 + \kappa_1 x_1 + \kappa_2 x_2 = 0$ any line on P. Then $(\kappa d) = 0$. The parameters of the residual points cut out of (1) by $(\kappa x) = 0$ are the roots of

$$(10) \qquad (\kappa a)t^2 + 3(\kappa b)t + 3(\kappa c) = 0$$

and (10) is apolar to (8), for

$$3(\kappa c)\beta + 3(\kappa a)\alpha' - 3(\kappa b)\beta' = 0,$$

as may be shown from (3) and the fact that $(\kappa d) = 0$.

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EXAMPLES OF A REMARKABLE CLASS OF SERIES.

BY PROFESSOR R. D. CARMICHAEL.

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Two-Fold and One-Fold Expression of the Properties of Functions.

1. IN the development of analysis during the past generation it has frequently happened that functions have arisen which are analytic in a sector of the complex plane and in

* Salmon, Higher Algebra, fourth edition, p. 180.