THE TWENTY-FOURTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE twenty-fourth annual meeting of the Society was held in New York City on Thursday and Friday, December 27–28, 1917, extending through two sessions on Thursday and a morning session on Friday. The attendance included the following forty-six members:

President R. J. Aley, Professor Joseph Bowden, Dr. T. H. Brown, Professor W. B. Carver, Professor F. N. Cole, Dr. G. M. Conwell, Dr. J. V. DePorte, Professor H. B. Fine, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Dr. G. M. Green, Professor C. C. Grove, Professor J. G. Hardy, Professor H. E. Hawkes, Dr. Olive C. Hazlett, Dr. T. R. Hollcroft, Professor L. A. Howland, Professor Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Professor P. H. Linehan, Professor C. R. MacInnes, Professor Helen A. Merrill, Professor R. L. Moore, Professor Frank Morley, Dr. G. W. Mullins, Dr. Alexander Pell, Professor Anna J. Pell, Dr. G. A. Pfeiffer, Professor H. W. Reddick, Professor R. G. D. Richardson, Dr. J. F. Ritt, Dr. Caroline E. Seely, Professor Clara E. Smith, Professor D. E. Smith, Professor Sarah E. Smith, Professor H. D. Thompson, Mr. J. N. Vedder, Professor C. W. Watkeys, Mr. H. E. Webb, Mr. R. A. Wetzel, Professor E. E. Whitford, Professor Ruth G. Wood, Professor J. W. Young.

At the opening session Professor R. G. D. Richardson took the chair. Professor J. W. Young presided at the following sessions. The Council announced the election of the following persons to membership in the Society: Dr. J. W. Campbell, Wesley College, Winnipeg, Canada; Dr. Mary F. Curtis, Western Reserve University; Mr. C. H. Parsons, Columbia University; Mr. J. B. Rosenbach, University of New Mexico; Mr. H. M. Terrill, Columbia University. Five applications for membership in the Society were received.

Committees were appointed to arrange for the summer meeting at Dartmouth College in 1918 and for the summer meeting and colloquium at the University of Chicago in 1919.

The total membership of the Society is now 735, including 77 life members. The total attendance of members at all meetings, including sectional meetings, during the past year was 351; the number of papers read was 167. The number of members attending at least one meeting during the year was 217. At the annual election 116 votes were cast. The Treasurer's report shows a balance of $9,762.98, including the life-membership fund of $6,333.13. Sales of the Society's publications during the year amounted to $1,474.19. The Library now contains 5,475 volumes, excluding unbound dissertations.

At the annual election, which closed on Friday morning, the following officers and other members of the Council were chosen:

Vice-Presidents,	Professor J. L. COOLIDGE, Professor D. R. CURTISS.
Secretary,	Professor F. N. COLE.
Treasurer,	Professor J. H. TANNER.
Librarian,	Professor D. E. SMITH.

Committee of Publication,

Professor F. N. COLE,
Professor VIRGIL SNYDER,
Professor J. W. YOUNG.

Members of the Council to Serve until December, 1920,

Professor R. C. ARCHIBALD, Professor D. N. LEHMER,
Professor DUNHAM JACKSON, Professor J. B. SHAW.

The following papers were read at the annual meeting:

(1) Professor F. L. HITCHCOCK: "The coincident points of two algebraic transformations."

(2) Professor W. B. CARVER: "The conditions for the failure of the Clifford chain."

(3) Professor C. J. KEYSER: "The rôle of the concept of infinity in the work of Lucretius."

(4) Professor C. J. KEYSER: "Concerning the number of possible interpretations of any system of postulates."

(5) Dr. W. H. WILSON: "Systems of functional equations which define hyperbolic sine, hyperbolic cosine, sine, and cosine uniquely."

(6) Dr. C. H. FORSYTH: "Tangential interpolation of ordinates among areas."

(7) Professor W. B. FITE: "Concerning the zeros of the solutions of certain differential equations."

(8) Professor R. L. MOORE: "Concerning a set of postulates for plane analysis situs."

(9) Dr. C. A. FISCHER: "Integral equations involving Stieltjes integrals."

(10) Professor O. E. GLENN: "Preliminary report on a new treatment of theorems of finiteness."

(11) Professor C. L. E. MOORE: "Rotations in hyperspace."

(12) Dr. G. M. GREEN: "Memoir on the general theory of surfaces and rectilinear congruences."

(13) Dr. J. F. RITT: "On the iteration of rational functions."

(14) Dr. OLIVE C. HAZLETT: "On rational integral invariants and covariants of the general linear algebra."

(15) Professor ANNA J. PELL: "Systems of linear equations."

(16) Dr. NORBERT WIENER: "Internal isomorphisms of complex algebra."

(17) Dr. T. R. HOLLCROFT: "A classification of general (2, 3) point correspondences between two planes."

(18) Professor W. F. OSGOOD: "Singular points of analytic transformations."

(19) Dr. M. T. HU: "Linear integro-differential equations with a boundary condition."

(20) Professor FRANK MORLEY: "Some general projective invariants of the algebraic planar curve."

The papers of Professor Hitchcock and Dr. Wilson were communicated to the Society through Professor C. L. E. Moore. Dr. Wiener was introduced by Professor Huntington. Dr. Hu's paper was communicated through Professor Bôcher. In the absence of the authors the papers of Professor Hitchcock, Dr. Wilson, Dr. Forsyth, Professor Glenn, Professor C. L. E. Moore, Professor Osgood, and Dr. Hu were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Let $x_1, x_2, \cdots, x_n$ be a set of n independent variables. For convenience of language we may say that these variables are the homogeneous coordinates of a point in space of $n - 1$

dimensions. Let $P_1, P_2, \cdots, P_n$ and $Q_1, Q_2, \cdots, Q_n$ be two sets of polynomials homogeneous in the variables x and of degrees p and q respectively, defining a pair of transformed points P and Q. There always exist one or more points x such that P and Q coincide. This is the same as saying that the two-row determinants of the matrix

$$\begin{matrix} P_1, P_2, \cdots, P_n \\ Q_1, Q_2, \cdots, Q_n \end{matrix}$$

vanish simultaneously for certain properly chosen sets of values of the variables $x_1, x_2, \cdots, x_n$. Professor Hitchcock shows that the number of such points is in general $p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + q^{n-1}$.

2. Clifford discovered a chainwise extension of the notion of the circumcircle of 3 lines, and showed* that n lines determine a circle when n is odd and a point when n is even. Professor Morley has given† analytic expressions for the Clifford circles and points in conjugate complex coordinates, and has shown that the Clifford circle becomes a straight line when a certain determinant vanishes. In the present paper Professor Carver shows that the Clifford chain fails when, and only when, the n lines are tangents to certain metric curves. These curves are rational, and of a type the dual (projectively) of the Jonquière curves.

3. Professor Keyser shows that the scheme of thought in the De Rerum Natura of Lucretius is dominated by the concept of infinity; that the author's conception of an infinite multitude accords with the current definition of the term; that the same is true of his notion of an infinite magnitude; that the infinites of Lucretius are not variables but are, like those of Cantor, static affairs; that, on the other hand, the Lucretian infinites are not composed of abstract things (points of space and pure numbers) but of concrete things (material particles, for example); that a Lucretian infinite multitude is always of the denumerable variety; and that the poet made repeated use of the characteristic whole-part property of infinite multitudes.

* "Synthetic proof of Miquel's theorem," Clifford's Mathematical Papers, p. 38 (1870).

† "On the metric geometry of the plane n-line," *Transactions Amer. Math. Soc.*, vol. 1 (1900), p. 97.

4. The main purpose of this note by Professor Keyser is to show that any postulate system involving at least one undefined term denoting an element as distinguished from a relation admits of any given infinite number of different interpretations.

5. In his Lehrbuch der Funktionentheorie, second edition, page 582, Professor Osgood has given a determination of the functions $\sin x$ and $\cos x$ on the basis of their addition theorems. In the present paper Dr. Wilson determines the functions $f(x) = \sin ax$ and $g(x) = \cos ax$ as the general continuous solutions of the simultaneous equations

$$g(x - y) = g(x)g(y) + f(x)f(y),$$

$$f(x - y) = f(x)g(y) - g(x)f(y),$$

where x and y are independent real variables and $f(x) \not\equiv 0$. It is furthermore shown that if x and y are not real and if $x = u + v\sqrt{-1}$, u and v real, then the general solutions continuous over the finite complex x-plane are

$$g(x) = \cos (au + bv), \qquad f(x) = \sin (au + bv),$$

the general solutions continuous along any line in the finite complex x-plane not parallel to the axis of reals are

$$g(x) = \cos bv, \qquad f(x) = \sin bv,$$

and the general solutions continuous along any line in the finite complex x-plane not parallel to the axis of imaginaries are

$$g(x) = \cos au, \qquad f(x) = \sin au,$$

where a and b are arbitrary constants. A similar determination of the functions $g(x) = \cosh ax$ and $f(x) = \sinh ax$ is made from the simultaneous equations

$$g(x - y) = g(x)g(y) - f(x)f(y),$$

$$f(x - y) = f(x)g(y) - g(x)f(y).$$

6. Dr. Forsyth modifies an interpolation formula used to interpolate ordinates when the data consist of areas so that in interpolating several ordinates in each of several successive intervals the discontinuities ordinarily occurring at the points

of intersection of any two consecutive interpolation curves are practically eliminated by requiring the slopes of these curves to be the same at the intersections.

7. In this paper Professor Fite generalizes the results of Kneser published in volume 42 of the *Mathematische Annalen* concerning certain linear differential equations whose solutions change sign an infinite number of times as the independent variable increases indefinitely through real values. He also establishes the existence of a minimum length for an interval within which all the functions which form a solution of a system of linear equations can vanish.

8. In his paper "On the foundations of plane analysis situs,"* Professor Moore exhibited three systems of axioms, Σ_1, Σ_2, and Σ_3. He proposes to show that every space that satisfies Σ_1 or Σ_2 is a number plane in the sense that it is in one-to-one continuous correspondence with euclidean space of two dimensions.

9. In this paper Dr. Fischer extends the definition of the Stieltjes integral $\int f(x)d\alpha(x)$ so that it is determined uniquely whenever both f and α have finite variation. It is then proved that when $f(x)$ and $K(x, y)$ have finite variation, and when the variation of $K(x, y)$ considered as a function of y alone is bounded, the equation

$$f(x) = u(x) + \lambda \int_a^b u(y)d_yK(x, y)$$

has a unique, finite solution, for sufficiently small values of λ. It is also proved that this equation can under certain conditions be put into the form

$$g(x) = u(x) + \lambda \int_a^b L(x, y)du(y),$$

and that this last equation has a solution for sufficiently small values of λ, when it is merely assumed that $g(x)$ has finite variation, that $L(x, y)$ is finite, and that when it is considered as a function of x alone its variation is bounded. Both the method of successive substitutions and that of reciprocal functions are employed.

* *Transactions*, vol. 17, no. 2 (April, 1916), p. 131.

10. The rationale of the method of Professor Glenn's paper is based upon a property, which he has established, that all concomitants of any one of several designated infinite systems are expressible as polynomials in certain elemental concomitants belonging to specified domains. Hilbert's theorem then applies.

The theory is adaptable to formal concomitants of a quantic to a prime modulus, concerning which further investigation is in progress.

11. Professor Moore first shows that a complex 2-vector in a space of $2m$ dimensions can always be resolved into the sum of m mutually completely perpendicular plane vectors. This resolution is not unique in case certain relations exist between products of these vectors. In four dimensions the resolution is not unique if the 2-vector is self-complementary. From these facts it is shown that a rotation in $2m$ dimensions can be resolved into rotations parallel to m mutually completely perpendicular planes. This resolution is not unique if the rates of rotation in two or more of the planes are equal.

12. In Dr. Green's paper certain geometric concepts introduced by him in previous communications to the Society are applied in the study of a curved surface and of certain associated rectilinear congruences. A line l in the tangent plane of the surface has related to it a line l' passing through the corresponding point of the surface, the lines l and l' being reciprocal polars with respect to the osculating quadric of that point. The lines l and l' are in the general relation R with respect to the asymptotic net of the surface, according to a definition introduced by Dr. Green in other papers.* A congruence Γ composed of lines l determines a reciprocal congruence Γ' of lines l'. By further specializing the relation between the congruences Γ and Γ' various important congruences may be defined which are uniquely determined by the surface. Such specialization of the relation between Γ and Γ' seems to be very fruitful both in the discovery of new congruences and in the characterization of known ones. Among these are the directrix congruences of Wilczynski, a pair of reciprocal congruences which reduce to Sullivan's

* Cf. in particular a recent note in the *Proceedings of the National Academy of Sciences*, vol. 3, no. 10 (Oct., 1917).

scroll directrix congruences when the surface is ruled, and the canonical congruences and pseudo-normal congruence introduced by Dr. Green at the summer meeting of 1916 and described in the note already cited.

A congruence Γ is said to be harmonic to a surface S if its developables correspond to a conjugate net on S, and a congruence Γ' is said to be conjugate to S if its developables correspond to a conjugate net on S. Characterizations are given for such congruences, and various questions are studied in connection with them. In particular, the characterization of planar nets with equal invariants is found to be closely related to the case in which the developables of the congruence Γ' are indeterminate.

The developables of a congruence Γ correspond to a net of curves on S, and the tangents to these curves are called Γ-tangents. The line l is met in its focal points by the tangents which are conjugate to the corresponding Γ-tangents, a property which has some important geometric consequences, in particular the geometric characterization of congruences harmonic to a surface.

13. Let $f_n(x)$ be the nth iterate of a rational function $f(x)$. The theorems presented by Dr. Ritt deal principally with the antecedents of any point a, that is, the points x such that, for some n, $f_n(x) = a$, and with the associates of a, the points x such that, for some n, $f_n(x) = f_n(a)$. The antecedents of a point are generally infinite in number, and the set of their limiting points contains a perfect set, which probably is an analytic curve; a similar statement holds for the associates. Perhaps the most interesting result given, part of which was found before by Professor A. A. Bennett, and which is an immediate consequence of the existence of certain meromorphic functions discovered by Poincaré, is the following:

If $f(a) = a$, and $|f'(a)| > 1$, then, given any point b, there is an antecedent of b in any neighborhood of a, however small.

It is shown also that in any neighborhood of such a point a there are an infinite number of points x such that $f_n(x) = x$ for some n.

14. Dr. Hazlett's paper proves some theorems on rational integral invariants and covariants for general linear algebras, analogous to well-known theorems in the classical invariant

theory for algebraic forms. For instance, such invariants must be homogeneous, and must possess a certain property which (by reason of the analogy with the classical invariant theory for algebraic forms) is called isobarism. Similarly for covariants. By the aid of these and other fundamental properties, it follows at once that the rational integral invariants are finite. In the cases $n = 1, 2$ all algebraic invariants and covariants are invariants and covariants respectively of the characteristic determinants. This is the converse of a theorem proved in a previous paper.

15. By means of an orthogonal system of linear forms and linear differential forms, Mrs. Pell shows that the theory of the system of linear equations

$$x_i = c_i + \sum_k a_{ik} x_k + \sum_\alpha \int_a^b \frac{da_i^{(\alpha)}(t)\, du^{(\alpha)}(t)}{da_0^{(\alpha)}(t)},$$

$$u^{(\alpha)}(s) = f(s) + \sum_k a_k^{(\alpha)}(s) x_k + \sum_\beta \int_a^b \frac{dA^{(\alpha,\beta)}(s,t)\, du^{(\beta)}(t)}{da_0^{(\beta)}(t)},$$

where $\{x_i\}$ and $\{u^{(\alpha)}(s)\}$ are unknown, is reduced to the theory of the linear equations

$$x_i = d_i + \sum_k b_{ik} x_k.$$

A special case is the theory of mixed linear integral equations. Existence and properties of solutions of a symmetric system are obtained.

16. An internal isomorphism of a system of elements and operations thereon is a one-to-one correspondence between the elements of the system and themselves which transforms operations belonging to the system into operations belonging to the system. Two such isomorphisms are considered as the same if they induce the same transformation of the operations of the system, independently of what they do to the elements. Dr. Wiener shows that the internal isomorphisms of the system in which the elements are the complex numbers and the operations are such that a polynomial in the operations and the operands with integral coefficients is zero, are all established by linear transformations.

17. A general (2, 3) point correspondence between two

planes (x) and (x') is said to exist when to any point of (x') correspond three points of (x) and to any point of (x), two points of (x') such that the correspondence in neither plane is involutorial. Such a correspondence is defined by pairs of algebraic equations in the coordinates (x_1, x_2, x_3) and (x_1', x_2', x_3') whose loci are systems of curves in the planes (x) and (x') intersecting in three and two points respectively. Each such system gives rise to a transformation relating curves of one plane to curves of the other and determines in each plane a locus of points two of whose image points in the other plane coincide, and likewise the loci of these coincident image points. The system is said to be independent if it cannot be reduced to another by any series of birational transformations. In all, Dr. Hollcroft finds twelve such independent systems and finally proves that any system defining a general (2, 3) point correspondence between two planes is birationally equivalent to some one of these twelve types.

18. A preliminary notice of Professor Osgood's paper appeared in the BULLETIN for June, 1917. The paper will appear in full in the *Transactions*.

19. Dr. Hu studies a linear integro-differential equation of a veiy general type with a linear boundary condition, and introduces such conceptions as adjoint systems and Green's functions.

20. After a suggestion for obtaining the discriminant, Professor Morley considers the degree of the conditions that a line meet the curve thrice at one point and twice at another, or twice at three points.

F. N. COLE,
Secretary.