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A CONSPECTUS OF THE MODERN THEORY OF DIVERGENT SERIES.*

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IT will doubtless be recalled by all here present that the first volume of the encyclopedia of mathematics, as originally published in German, was issued in hefts, or parts, each prepared by a special collaborator well qualified for his task. The completion of the volume occupied a period of about six years and resulted in a finished work of no less than 1,128 pages. This was followed some years later by the second volume, which was still larger, containing in all its parts 1,154 pages. The two volumes taken together were intended to cover the entire range of pure mathematical analysis (as distinguished from geometry, mechanics, and other applications), each important branch being treated in its essentials and the amount of space allotted to any one topic being proportionate, at least roughly, to its relative importance. Out of the grand total of 2,282 pages thus presented it may now be remarked that a little less than 7 pages were devoted to that particular topic specified as "divergent series." Thus we have a ratio of 7 to 2,282, or about three tenths of one per cent, which may fairly be taken as the measure of interest in this topic at the time when the encyclopedia began to appear; that is, in the neighborhood of twenty years ago. The 7 pages in question, as we come to examine them, seem directed mainly to showing by means of simple illustrations that the processes by which Euler arrived at certain noteworthy results while

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dealing with divergent series are in themselves altogether unjustified and unscientific, the correctness of his results being in the nature of a happy circumstance arising out of the inherent character of the particular series he happened to be dealing with. On the whole, the short article conveys a rather gloomy outlook for this entire field of study, especially as regards any attempts that may be made to put it upon a truly scientific basis. Without digressing further upon the attitude of the encyclopedia, I take for granted that to-day we are all quite willing to agree that, contrary to any predictions that may have been made in the past, divergent series have now come to occupy a prominent place in analysis and one that bids fair to be permanent. How, it may be asked, has this happened? What has been done during these twenty years that really constitutes a vital advance in this field of study? Can we say that divergent series are at last upon a scientific basis? These are fair questions and it is to them that I would respectfully direct your attention for a few moments this afternoon. In answering, I shall attempt no more than an outline or conspectus of the situation as I myself have come to regard it and in this I am conscious beforehand that my own feelings, at least at some points, may not be universally acceptable; yet I shall venture all with equal candor. I shall not attempt to detail a large body of more or less intricate theorems and results, but I shall endeavor in a general way to show what seems central to me both as to the logical position of the present day theory and its applications.

In the first place, if I may refer again to Euler, we should of course recognize clearly at the outset that in his day the class of series which we now call divergent had not been clearly separated off by itself and in this sense it really had no well-defined meaning. It is true that certain individuals of the class had been studied more or less extensively and in ways which we are now bound to regard as interesting because of their curiosity, as for example the assignment of the sum $\frac{1}{2}$ to the oscillating divergent series

$$1 - 1 + 1 - 1 + 1 - 1 + \; \ldots,$$

but divergent series as a class had not been carefully defined. Not until the time of Cauchy and Abel did they take on an exact sense and it was then through a purely negative process.

In fact, they were then defined, as they still are to-day, as being all those series which do *not* satisfy the particular definition for convergent series which was formulated by these two great mathematicians. On this plan, convergence means nothing more or less than that the sum of the first n terms approaches a limit as n increases indefinitely, while divergence means simply that this particular limit does not exist. However natural this sort of a classification of series may seem to us, owing to our having been born and brought up with it, we must recognize that it involves after all a large element of arbitrariness. In fact, the only reason why the fundamental distinction between series should be made to hinge upon the existence of this particular limit lies in the fact that those series for which it exists are an admittedly important class. Of course there is no adequate reason in this to account for the way in which the excluded or divergent class were so long held in utter disrepute by the successors of Abel and Cauchy and continue still to repel us, except it be the psychological fact that whatever is excluded from a certain good class we instinctively think of as forming a bad class. However, to banish a whole class of series from analysis simply because it does not bear the stamp of a certain branded variety may easily be and is too hasty and rough a handling of their case. It makes no allowance for the different degrees of respectability which, though divergent, they may still possess. Thus it is, in substance, that the modern studies on divergent series may be said to have arisen. They are a reaction against the underlying arbitrariness of the Abel-Cauchy distinction and the ruthless and unjustified exclusions which this particular distinction has tended to produce. If we inquire just what the avenue of approach has been to this subject, it may be said that the first tangible result was the creation of the so-called "sum formulas," the essential feature of any such formula being that it shall not only serve, in case it can be evaluated, to give us the sum of any convergent series, but it shall continue to preserve a meaning when applied in the same way to certain divergent series, thus associating, or assigning, definite numerical values or sums to them also. The spirit of this procedure is, of course, nothing more or less than that common to all extensions of idea in mathematics. Just as in the theory of functions of a complex variable the sine of x, or any other function such as there considered, comes to have a meaning for complex values of x only by virtue of a certain definitional formula so constructed as to give results when x is real that are in accord with the known properties of the sine in the real domain, so in the modern studies on divergent series sums are assigned to such series only by virtue of sum formulas so constructed as to yield results when applied to convergent series that are in accord with the known facts regarding such series. I have purposely mentioned this particular illustration because by carrying it one step farther, as I shall now do, I shall be able to bring out another feature of the present day situation as regards divergent series, and this time we shall discover that the theory is by no means in a perfectly satisfactory logical state as yet, being in this respect quite different from any well established body of doctrine such as the theory of functions of a complex variable. In fact, it will be recalled that it is a vital feature of the complex variable theory that no two different definitions for any one given function, as sin x, are possible. In other words, it is demonstrable that if any two definitions for the function agree throughout the real domain, they will necessarily agree also throughout the complex domain. This property of uniqueness, if we may so designate it, is brought about by imposing a fundamental limitation at the very outset of the theory, namely, that only those functions shall be retained which belong to the class known as monogenic, or analytic. Thus, the logical coherence of the complex variable theory is bought at the price of a far reaching initial restriction, yet one which, as we know, is not so serious but what we are left with a class of functions of great interest in themselves and in their applications. In the theory of divergent series, on the other hand, as it exists today we are in possession of a large number of sum formulas each applicable in the sense before described to a divergent series and all agreeing with each other so far as convergent series are concerned, but *not* agreeing in general in the sums they prescribe to a given divergent series. In a word, the situation is analogous to that which the theory of functions would present in case the fundamental limitation as to the character of the functions considered were to be removed. How, you will at once inquire, can there be any general theory worthy of the name on such a loose plan as this? Strictly speaking, there cannot. However, all that is lacking in order that we have a bona fide theory is that we come to some agreement, as in the theory of functions of a complex variable, as to what is proper by way of initial limitations, either upon the kind of divergent series to be considered or upon the kind of sum formulas we shall employ in connection with them. Have we any good indications as to what such limitations may well be? Yes, but none that are universally accepted as yet. As to what the line of approach to this matter may properly be I shall have more to say eventually, but for the moment I deem it more proper that I attempt to answer another question which no doubt by this time has arisen in your own minds. How can it be that divergent series have come into prominence in the face of such conditions as I have just described, whereby there does not exist even to this day a strictly coherent and universally acceptable general theory of the subject? This is a very proper question, whose answer is twofold.

First, the various sum formulas, regardless of their interrelations and other such logical aspects, have been found to yield interesting information when applied to certain important special series, such as Fourier's series, Dirichlet's series, etc. For example, even though the Fourier series representing a given function $f(x)$ may be divergent at a certain point, still the series may be summable there by one or more formulas and the sum thus obtained may and usually does continue to serve fully as useful a purpose from the standpoint of mathematical physics or other applications as does the sum when the same series is convergent; that is, the sum in the extended sense furnishes the answer to the proposed physical problem. Extensive investigations have been carried out in this connection to determine sufficient conditions under which a Fourier series will be summable, analogous to the well-known conditions for its convergence, and similarly the problem has been carefully worked over for some of the other related developments such as those for an arbitrary function in terms of Bessel functions, or Legendre functions. In this connection it may be of interest to observe the central fact that whereas the Fourier series for $f(x)$ converges in general to the value

$$\frac{f(x-0)+f(x+0)}{2}$$

only in case $f(x)$ is of limited total fluctuation in the neighborhood of the point x under consideration, the same series will be summable by the simplest of the well known sum formulas of Cesàro to this same value provided only that the right and left limits, namely $f(x+0)$ and $f(x-0)$, exist. Here we incidentally meet with an illustration of the manner in which summability includes convergence as a special case, since the latter set of conditions is evidently much less restrictive than those just mentioned for convergence. And it may be added that, so far as the Bessel expansions are concerned, much the same situation prevails as with Fourier series, but the case of the Legendre expansions is essentially different and all the more interesting because of its novelty. Here no new results follow so long as one uses the simplest of the Cesàro formulas; that is, so long as one uses the formula of order 1 these developments are summable under no less restrictive conditions than would insure convergence itself. But by using the same formula of order 2, or 1 higher, various new and interesting results follow. The manner in which the summable properties of the Legendre developments thus lie intermediate between the range of the formulas of orders 1 and 2 has led, it may be added, to a generalization of the whole conception of the formula to include fractional or even incommensurable orders of summation. Out of such a generalization there arise interesting special studies analogous to what we find in the ordinary elementary study of series. Thus, just as we know that in the case of the series

$$1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots$$

convergence merges into divergence as p passes through the value 1 from above, so we are able to determine for divergent series just where the critical order, or orders, are at which new information begins to be realized. Furthermore, the extended study of sum formulas simply upon their own merits, that is, without stressing their logical interrelations, has led very naturally to the notion of uniform summability corresponding to that of the uniform convergence of a series. And again, alongside of the same studies, corresponding studies have naturally arisen for divergent integrals. Here as before the underlying idea is that of setting up a formula which shall give the value of all convergent improper integrals and at the same time preserve a meaning and thus assign a value to some integrals that are divergent. As the formulas pertaining to such studies are not as familiar as those for series, it may be proper to note that if the type integral be taken as

$$I = \int_a^\infty f(\beta)d\beta,$$

then the formula analogous to the Cesàro formula of order 1 and giving the value of I even in some cases of divergence is

$$I = \lim_{x=\infty} \frac{1}{x} \int_a^x d\alpha \int_a^x f(\beta)d\beta,$$

this formula having been first obtained, I believe, by Professor C. N. Moore in his thesis. It is equivalent to the somewhat simpler form

$$I = \lim_{x=\infty} \int_a^x f(\beta)\left(1 - \frac{\beta}{x}\right)d\beta.$$

I shall not attempt any further details concerning sum formulas and the facts derivable either directly or by suggestion from them. We can certainly say, however, that this class of studies, as carried out independently for the various formulas, has greatly increased the range of interest in series in general and has enabled us to see convergent series in particular from new and very instructive points of view.

The second reason alluded to above for the prevalent interest in divergent series despite the lack as yet of any universally accepted general theory about them brings us to a certain very important aspect of the whole which we have not as yet mentioned—an aspect, it may be added, which was entirely disposed of in 11 lines of the encyclopedia article mentioned at the beginning. We refer to what is known as "asymptotic series." This is in reality the oldest aspect which our subject presents. It may be said to have originated in an isolated note by Cauchy in 1843 relating to the well-known series of Stirling

$$\begin{aligned}\log \Gamma(x) = \tfrac{1}{2}\log 2\pi + (x - \tfrac{1}{2})\log x - x \\ + \frac{B_1}{1\cdot 2}\frac{1}{x} - \frac{B_2}{3\cdot 4}\frac{1}{x^3} + \frac{B_3}{5\cdot 6}\frac{1}{x^5} - \cdots\end{aligned} \tag{1}$$

$$(B_m = m\text{th Bernoulli number}).$$

Cauchy pointed out that this series, though divergent for all values of x, may be used in computing $\log \Gamma(x)$ when x is large (and positive). In fact, he showed that, having fixed the number n of terms taken, the absolute error committed by stopping the summation at the nth term is less than the absolute value of the next succeeding term, and hence becomes arbitrarily small ($n > 3$) as x increases indefinitely. Cauchy's work on divergent series was confined, however, to the single series (1) and, owing to the overemphasis placed upon convergent processes by the successors of Cauchy and Abel, as mentioned earlier, no further progress was made in this field until the subject at last reappeared after more than forty years in conneetion with the researches of Poincaré upon the irregular solutions of linear differential equations. Poincaré considered those divergent series (normal series) of the form

$$\begin{aligned} &e^{f(x)}x^{\rho}(A_0 + A_1/x + A_2/x^2 + \cdots); \\ &f(x) = \text{a polynomial in } x, \qquad \rho = \text{a constant}, \end{aligned} \tag{2}$$

which for some time had been known to satisfy *formally* homogeneous linear differential equations of certain types having the point $x = \infty$ as a so-called "irregular point," and he showed essentially that in general to every such formal solution there corresponds an *actual* solution which can be represented by (2) in much the same sense as (1) was described above as representing $\log \Gamma(x)$. In view of the important significance of such results both from the standpoint of the possible use of divergent series and from that of the theory of differential equations, Poincaré set apart and discussed in some detail a broad class of divergent series of the special form (2), applying to them the name of "asymptotic series." Poincaré's results, however, in so far as they concerned differential equations, were noticeably incomplete, being limited by certain unfortunate restrictions, and thus his original studies have given rise in later years to numerous researches in which noteworthy advances have been made, though open questions in this connection still remain. Corresponding investigations, likewise begun by Poincaré, pertaining to linear difference equations have also been undertaken in recent years and carried to an advanced stage. Meanwhile another important aspect of the theory of asymptotic series has come into view; namely, that of actually determining the asymptotic developments of any given function—a problem of decided interest for the study and classification of functions in general. I can perhaps employ the next few moments to no greater advantage than in sketching somewhat more completely these various fundamental problems relative to asymptotic series, pointing out incidentally certain unsolved special problems of noteworthy interest.

Referring again to differential equations, the studies in question may be said to center about the linear homogeneous differential equation

$$(3)\quad y^{(n)} + a_1(x)y^{(n-1)} + a_2(x)y^{(n-2)} + \cdots + a_n(x)y = 0,$$

wherein the coefficients $a_1(x)$, $a_2(x)$, $\cdots$, $a_n(x)$ are, in the simplest case, rational functions of x and, in the more extended case, are supposed to be developable in series of the form

$$(4)\quad a_r(x) = x^{rk}\left[a_{r,0} + \frac{a_{r,1}}{x} + \frac{a_{r,2}}{x^2} + \cdots\right] \quad (r=1,2,\cdots,n),$$

k being zero or a positive integer. In such a differential equation the point $x = \infty$ is in general an irregular point, so that the usual normal solutions are divergent series of the form (2). With reference to these solutions, we may now cite the following fundamental theorem:

"If for the equation (3) the roots m_1, m_2, m_3, $\cdots$, m_n of the so-called characteristic equation, i. e., of the algebraic equation

$$(5)\qquad m^n + a_{1,0}m^{n-1} + a_{2,0}m^{n-2} + \cdots + a_{n,0} = 0,$$

are distinct, equation (3) possesses n linearly independent solutions y_1, y_2, y_3, . . . , y_n which, for large values of x, are developable asymptotically in the form (2), wherein $f(x)$ is of degree $k+1$, and $a_0 = 1$; that is, we have

$$(6)\quad y_r \sim e^{f_r(x)}x^{\rho_r}\left[1 + \frac{A_{1,r}}{x} + \frac{A_{2,r}}{x^2} + \cdots\right] \quad (r=1, 2, 3, \cdots, n),$$

where $f_r(x)$ is a polynomial of degree $k+1$ in x, while ρ_r is a constant."

If in this theorem the restriction be removed that the roots of the characteristic equation be distinct; that is, if multiple roots are present, the theorem fails and we at once encounter a problem for which no general solution has yet been obtained. Moreover, the theorem as just stated carries with it the assumption that x is real. When x is regarded as complex, much the same results follow, the forms (6) holding now over certain sectors of the plane emanating from the origin, but here again much remains to be investigated, the existing theorems covering only what may be described as the simplest cases.

Contrasted with the same theorem for differential equations, is the corresponding fundamental theorem for linear difference equations:

"Given a homogeneous linear difference equation of the nth order, which we may write in the form

$$(7) \qquad \begin{aligned} y(x+h) + a_1(x)y(x+h-1) + a_2(x)y(x+h-2) \\ + \cdots + a_n(x)y(x) = 0, \end{aligned}$$

and let it be assumed that the coefficients $a_1, a_2, \cdots, a_n$ are either rational functions of x or are developable in series of the form (4). Then, if the roots $m_1, m_2, \cdots, m_n$ of the characteristic equation (5) are distinct and no one of them equal to zero, equation (7) possesses n linearly independent solutions $y_1, y_2, \cdots, y_n$ valid for large positive values of x and developable asymptotically in the forms

$$(8) \qquad y_r \sim [\Gamma(x+1)]^k m_r^x x^{\rho_r}\left[1 + \frac{A_{1,r}}{x} + \frac{A_{2,r}}{x^2} + \cdots\right] \qquad (r = 1, 2, 3, \cdots, n).$$

In case the characteristic equation presents multiple roots, or a zero root, no corresponding results appear to have been obtained, at least in general, though this whole subject has been interestingly discussed from an altogether different point of view and in a considerably larger measure of completeness by the introduction throughout of the so-called faculty series instead of the usual power series forms. Here again much remains to be done for the case of a complex variable, though a beginning corresponding to that cited above for differential equations has been made. The importance of these studies, both as regards differential and difference equations, lies, of course, in the fact that it is equations of these particular types that play a most fundamental rôle in analysis, both from the function theoretic standpoint and from that of applications. We shall not enter, however, into further details in this direction more than to mention the fact that corresponding studies for non-homogeneous differential and difference equations have been considered but, like the homogeneous cases, are in but a limited state of completion.

As regards the problem of determining the asymptotic developments of a given function, which I also mentioned a moment ago, the meaning of this class of studies may perhaps be best understood from one or two simple illustrations. Let us take, for example, the following power series in which x is regarded as taking complex as well as real values:

$$(9) \qquad f(x) = \sum_{n=0}^{\infty} \frac{x^n}{\rho - n}; \quad \rho = \text{any non-integral constant.}$$

The radius of convergence of this series is easily seen to be 1, so that the series itself yields no information as to the nature of the function $f(x)$ defined by it in the more distant portions of the plane. In order to secure such information and thus be able to follow the course of the function for values of x of large modulus it becomes necessary to develop $f(x)$ in some manner about the point infinity, as for example in power series in $1/x$, but the simple knowledge of the formula for the nth term of the given series (9) provides no immediate way of determining the coefficients of such a development. When once obtained, moreover, it may either converge or it may represent $f(x)$ only asymptotically. How actually to determine the development, whatever be its ultimate character, is the problem, and in the case before us it may be stated that it becomes

$$f(x) \sim -\frac{\pi(-x)^\rho}{\sin \pi\rho} - \frac{1}{(\rho+1)x} - \frac{2}{(\rho+2)x} - \cdots,$$

this form holding at least so long as we confine ourselves to any sector of the plane which does not contain the positive half of the real axis. More generally, it may be shown that if we have any power series

$$\sum_{n=0}^{\infty} g(n)x^n,$$

wherein the coefficient $g(n)$ may be regarded as a function of a complex variable n and as such is analytic throughout the entire n plane except for a finite number p of poles, and at the same time, when considered for values of n of sufficiently large modulus, remains less than a constant, then the function $f(x)$

defined by this series will be developable in general throughout the distant portions of the plane either asymptotically or in a convergent series of the specific form

$$f(x) \sim -\sum_{m=1}^{p} r_m - \frac{g(-1)}{x} - \frac{g(-2)}{x^2} - \frac{g(-3)}{x^3} - \cdots,$$

where r_m represents the residue of the function $\dfrac{\pi g(n)(-z)^n}{\sin \pi n}$ at the rth pole of $g(n)$. Corresponding results for various other type forms of power series have likewise been obtained, and again, similar studies have been extensively carried out for functions defined not by power series, but by altogether different though very important forms, such as infinite products, or faculty series. Sufficient has been said, I judge, on this aspect of asymptotic series so that you will perceive its bearing not only upon the determination of the values of a function in distant regions but also upon the broader problem of the classification of functions in general, since functions may clearly be distinguished from one another in classes corresponding to the different characters of their asymptotic developments. Very much remains to be done in this entire field of investigation.

What we have thus far said may be briefly summarized in the statement that the modern theory of divergent series contains essentially two branches, the first concerning the question as to how a sum may be assigned to a divergent series in general, and the second pertaining merely to the functional properties of that important special class of divergent power series known as asymptotic series. Of these two branches, the second, though characterized by theorems and results which usually bear a high degree of complexity, presents no logical inconsistencies and is thus in quite as satisfactory a state as convergent series themselves, while the first, or problem of summation, when considered as a whole is still in an unsatisfactory logical state because, as pointed out earlier, we have a large variety of sum formulas which, though agreeing with one another when applied to convergent series, fail to do so to a greater or less degree when applied to divergent series. However, all that remains in order to bring about perfect agreement everywhere is, as was also stated earlier, that we place proper limitations either upon the kind of divergent series to be considered, or upon the kind of sum formulas to be retained, or both. In the time that remains to me I may therefore return for a few moments to the question as to how a logically consistent general theory of summation, if there is to be one, may well be constructed. It may be that the limitations which I am about to place may seem too restrictive to some, yet I take for granted that everyone shares with me the instinctive feeling that there should be a logically sound and fairly useful general theory of *some* sort and it is mainly in that spirit that my suggestions will be made.

In approaching the question let us first cast a glance back over the historical genesis of all the various sum formulas, for this is at once suggestive. The earliest and simplest of them is the one growing out of certain studies of Frobenius in 1880 relative to the behavior of the power series

$$\sum_{n=0}^{\infty} a_n x^n \tag{10}$$

for values of x *upon* its circle of convergence. His theorem in substance was as follows: "Suppose that the radius of convergence of (10) is 1, and let $s_n = a_0 + a_1 + a_2 + \cdots + a_n$. Then, we shall have

$$\lim_{x=1-0} \sum_{n=0}^{\infty} a_n x^n = \lim_{n=\infty} \frac{s_0 + s_1 + s_2 + \cdots + s_n}{n+1} \tag{11}$$

whenever the limit on the right exists." This is a straightforward result in the theory of functions having at first sight no relation to divergent series, nor indeed did it come to play any recognized part in the development of the latter for a considerable time. When it did it was because the left member of (11) is known to be identical with

$$\sum_{n=0}^{\infty} a_n \tag{12}$$

whenever this series is *convergent*, so it seemed natural in case it was divergent to continue assigning sums s to it in accordance with the formula

$$s = \lim_{n=\infty} \frac{s_0 + s_1 + s_2 + \cdots + s_n}{n+1} \tag{13}$$

so long as the limit on the right exists. This particular formula for s found additional justification in the fact that the value it assigns to (12) when divergent is useful in the sense that it is the value of the function $f(x)$ defined by (10) at the point $x = 1$; that is, formula (13) furnishes the analytic continuation of the power series (10) at the point $x = 1$ upon its circle of convergence. It was essentially in this same spirit of useful as well as possible extension of idea that the sum formulas of Cesàro and Hölder, both of which contain (13) as a special case, were set up, as likewise the later transcendental sum formula of Borel wherein a definite integral is involved. These considerations immediately suggest that a logically coherent and at the same time useful theory of summability may be established by limiting ourselves throughout to those series (12) for which the corresponding power series (10) has a non-vanishing radius of convergence and furthermore limiting our use of sum formulas to those which, like the familiar ones of Cesàro and Borel, assign to a divergent series (12) a sum s which is equal to the value of the analytic continuation of this same power series (10) at the point $x = 1$. Such a value for s allows of no duplicity and is therefore unique, thus removing the primary logical defect heretofore mentioned in the present day aspect of the theory. Moreover, a theory thus limited in scope at once satisfies our desideratum of being useful, for it attaches itself in a most fruitful way to the very important subject of analytic continuation in the theory of functions of a complex variable. For example, it may be shown that in general any divergent power series is summable by Cesàro's formula at points upon its circle of convergence, thus furnishing the analytic continuation of the corresponding function at such points, but that it is not summable by this formula at points outside the same circle. On the other hand, if Borel's integral formula be used on the same series, it gives a sum and hence the analytic continuation not only for points upon the circle of convergence, but in certain regions lying outside this circle; namely, within the so-called polygon of summability formed by tangents to the circle at those points which are singular points of the function defined by the corresponding power series. Moreover, in a theory as thus restricted the usual rules for the manipulation and combination of convergent series are in large measure preserved. In short, we have left, it seems to me, a sufficient body of doctrine to be worthy of the name "general theory of summation."

I trust that I have now made clear my own feelings regarding the three questions raised at the outset; first, as to why divergent series have come into such prominence since the appearance of the early volumes of the encyclopedia, second, what has been done that really constitutes a vital advance and third, as to whether such series are at last upon a truly scientific basis. My only fear is that in attempting to couch the whole in very simple form I may have gone too far in this direction and thus violated a principle which, I believe it is said, the poet Browning always carefully observed; namely, of never using so simple a style that the intelligence of one's readers or hearers may be offended. But this is a rather treacherous principle, as most people discover in attempting to read Browning, so I may perhaps be pardoned if I have seemed to depart too far from it.

SOLUTIONS OF DIFFERENTIAL EQUATIONS AS FUNCTIONS OF THE CONSTANTS OF INTEGRATION.

BY PROFESSOR GILBERT AMES BLISS.

(Read before the American Mathematical Society December 29, 1917.)

THE purpose of this note is to prove the differentiability of the solutions of a system of differential equations with respect to the constants of integration by a method which seems more natural and simpler than those which have hitherto been published. Incidentally a restatement of the so-called "imbedding theorem" for differential equations is given, a theorem which is frequently applied in the calculus of variations, and which has been useful, and could be made still more so, in many other connections. It is analogous to the fundamental theorem for implicit functions in its statement that a solution of a system of differential equations given in advance is always a member of a continuous family of such solutions.

Let C be an arc

$$(C) \qquad x = u(\tau), \qquad \tau_1 \leqq \tau \leqq \tau_2,$$