Multiplying (15) by \( m \), we have

\[ C m \cdot n = C a_{12} = 0. \]

Since \( a_{12} \neq 0 \), \( C = 0 \). Likewise multiplying by \( n \) we see that \( B = 0 \). Hence equation (14) becomes

\[ A \frac{\partial^2 Z}{\partial u \partial v} = 0. \]

Hence, the minimum surface is a surface of translation. The necessary and sufficient condition that a surface in hyperspace be a minimum surface is that the minimum lines on it are characteristics.

Massachusetts Institute of Technology.

---

SOME ALGEBRAIC CURVES.

By Dr. James H. Weaver.

(Read before the American Mathematical Society, April 28, 1917.)

In the following paper two algebraic curves are set up and some of their singularities are discussed. The author believes them to be new. At least a search through considerable of the literature on curves has failed to reveal them.

I.

Let there be any two distinct points \( A \) and \( B \). Let the line joining \( A \) and \( B \) be drawn, and let the distance \( AB = c \). Let there be drawn through \( A \) a line \( l_1 \) making an angle \( \theta \) with \( AB \), and let there be drawn through \( B \) a line \( l_2 \) making an angle \( n\theta \) with \( AB \) (\( n \) an integer). We also consider that \( AB \), \( l_1 \), and \( l_2 \) are in one plane. Let the intersection of \( l_1 \) and \( l_2 \) be \( C \). It is required to find the locus of \( C \).

Let \( A \) be the origin and let \( AB \) be the \( x \)-axis. Then the equations of the lines \( l_1 \) and \( l_2 \) will be

\[
\begin{align*}
(1) \quad y &= x \tan \theta, \\
(2) \quad y &= (x - c) \tan (n\theta)
\end{align*}
\]

respectively.
After eliminating $\theta$ from (1) and (2) we get

$$x^n - \binom{n}{2} x^{n-2} y^2 + \binom{n}{4} x^{n-4} y^4 - \cdots + (-1)^{(n-1)/2} xy^{n-1}$$

(3)

$$= (x-c) \left[ \binom{n}{1} x^{n-1} - \binom{n}{3} x^{n-3} y^2 + \cdots + (-1)^{(n-1)/2} y^{n-1} \right],$$

where $n$ is of the form $2k + 1$. A similar form holds if $n$ is of the form $2k$. The theory is the same in either case. (3) is then the equation representing the locus of $C$. Let us call this curve $C_n$. If in (2) we replace $n$ by $n - r$, we obtain for (3) a curve of degree $n - r$, which we will call $C_{n-r}$. ($r = 1, 2, \ldots, n - 1$)

From (3) it is evident that $C_n$ has an $(n - 1)$-point at the origin. The equations of the $n - 1$ tangents at this point will be given by

$$\left( \binom{n}{1} x^{n-1} - \binom{n}{3} x^{n-3} y^2 + \cdots + (-1)^{(n-1)/2} y^{n-1} \right) = 0.$$  

(4)

The factors of (4) are

$$y - x \tan \left( k\pi/n \right) \quad (k = 1, \ldots, n - 1).$$

Therefore the tangents to $C_n$ at the point $A$ together with the $x$-axis divide the angular magnitude about $A$ into $2n$ equal parts.

We will now consider the relation of the fixed point $B$ to the curve $C_n$. Let us write (3) in homogeneous coordinates. It will then be

$$x^n - \binom{n}{2} x^{n-2} y^2 + \cdots + (-1)^{(n-1)/2} xy^{n-1}$$

(5)

$$= (x - cz) \left[ \binom{n}{1} x^{n-1} - \cdots + (-1)^{(n-1)/2} y^{n-1} \right].$$

The first polar of this curve with respect to the point $B = (c, 0, 1)$ is

$$x^{n-1} - \binom{n-1}{2} x^{n-3} y^2 - \cdots + (-1)^{(n-1)/2} y^{n-1}$$

(6)

$$= (x - cz) \left[ \binom{n-1}{1} x^{n-2} - \cdots + (-1)^{(n-2)/2} y^{n-2} \right].$$
This process may evidently be continued. We may then state the following

Theorem: The $r$th polar of $B$ with respect to $C_n$ is $C_{n-r}$.

II.

Again let there be three distinct points $A$, $B$, and $C$ on the same straight line $l$, and through the point $C$ let the line $l_1$ be drawn perpendicular to $l$. Let lines $l_2$ and $l_3$ be drawn through $A$ and $B$ respectively, and let $l_2$ and $l_3$ intersect on $l$. Let $l_2$ make an angle $\alpha$ with $l$, and $l_3$ make an angle $\beta$ with $l$, and let a line $l_4$ be drawn through $B$, making an angle $n\beta$ with $l$. Let $l_2$ and $l_4$ intersect in $D$. Then just as in section I, the equation representing the locus of $D$ is

\[
k \left[ x^n - \binom{n}{2} x^{n-2} y^2 + \cdots \right]
\]

\[(7)\]

\[= (x - c) \left[ \binom{n-1}{1} x^{n-1} - \binom{n-3}{3} x^{n-3} y^2 + \cdots \right],\]

where $k = (a - c)/a$ and $a = AC$, and $c = AB$.

It is then evident that the theorem in section I holds for the curve represented by equation (7).

Ohio State University.

ON THE RECTIFIABILITY OF A TWISTED CUBIC.

BY DR. MARY F. CURTIS

(Read before the American Mathematical Society, April 27, 1918.)

Given the twisted cubic

\[(1) \quad x_1 = at, \quad x_2 = bt^2, \quad x_3 = ct^3, \quad abc \neq 0;\]

to show that the condition that it is a helix is precisely the condition that it is algebraically rectifiable.

If (1) is a helix, then $T/R$, the ratio of curvature to torsion, is constant. Denoting differentiation with respect to $t$ by