he did in others. His Algebra, for example, affords numberless instances in point.

In the early years of our professional lives we were in constant intercourse over such matters. Each of us was seeking to clarify and simplify his subject. Neither of us regarded the theory of functions of a real or of a complex variable as an end in itself, for each had his own ulterior uses for the theory—Bôcher, his differential equations, both complex and real. In fact, for each of us the theory of functions was *applied mathematics*, and in presenting its subject matter and its methods to our students, our aim was to show them great problems of analysis, of geometry, and of mathematical physics which can be solved by the aid of that theory.

Bôcher was quick to grasp the large ideas of the mathematics that unfolded itself before our eyes in those early years. His attitude toward mathematics helped me to have the courage of my convictions. The Funktionentheorie is largely Bôcher's work, less through the specific contributions cited on its pages than through the influence he had exerted prior to 1897—long before a line of the book had been written. We worked together, not as collaborators, but as those who hold the same ideals and try to attain them by the same methods. It was constructive work, and in such Bôcher was ever eager to engage.

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A THEOREM ON LINEAR POINT SETS.

BY DR. HENRY BLUMBERG.

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LET A be any given linear point set. We define the "relative exterior measure* of A in the interval I" as $m_e\ (A, I)/l$, where $m_e\ (A, I)$ represents the exterior measure (Lebesgue) of the subset of A in I, and l is the length of I. We then define the "relative exterior measure of A at the point x" as

* Cf. Denjoy, *Journal de Mathématiques*, ser. 7, vol. 1 (1915), p. 130.

k, if

$$\lim_{l_n \to 0} \frac{m_e(A, I_n)}{l_n} = k$$

for every sequence $\{I_n\}$ of intervals enclosing x and having a length l_n that approaches 0 with increasing n. It is the purpose of this note to prove the

THEOREM. *The relative exterior measure of every linear point set exists and is equal to* 1 *at every one of its points, with the possible exception of those of a set of measure* 0.

Proof. Let M represent the subset of the given set A that consists of points where the relative exterior measure of A is not 1. Hence there exists, for every point x of M, an integer n_x such that for every $\epsilon > 0$ there is an interval enclosing x and of length $< \epsilon$ in which the relative exterior measure of A is $< 1 - 1/n_x$. For a given x, let n_x' be the smallest integer having the property just described. Designating by M_n the set of x's for which $n_x' = n$, we obtain the decomposition

$$M = M_2 + M_3 + \cdots + M_n + \cdots.$$

Our theorem will be established if we show that M_n is of measure zero.

Let I, of length l, be any interval of the linear continuum. According to the definition of M_n, every point x of M_n may be enclosed in an arbitrarily small interval J_x in which the relative exterior measure of M and a fortiori of M_n is $<1-1/n$. The intervals J_x may be so chosen that they lie entirely in an interval I' of length $< l + \delta$, where δ is any given positive number. According to a well known theorem, we may select from the J_x's a denumerable infinity $[J', J'', J''', \cdots]$ having the same interior points as all the J_x's. Let m_n be the measure of the portion of I' covered by the intervals $J', J'', \cdots, J^{(n)}$. We distinguish two possibilities:

$$\lim_{n\to\infty} m_n \leqq \tfrac{2}{3}l \tag{1}$$

and

$$\lim_{n\to\infty} m_n > \tfrac{2}{3}l. \tag{2}$$

We shall prove that in either case the exterior measure of the subset of M_n in I is $\leqq [1 - (1/3n)]l$. In the first case, this is evident. In the second case, suppose $m_\nu > \frac{2}{3}l$. It is easily shown by elementary reasoning that we may extract from the sequence $J', J'', J''', \cdots, J^{(\nu)}$ a sequence $\bar{J}', \bar{J}'', \cdots, \bar{J}^{(\bar{\nu})}$ which cover the same portion of I' and no part more than twice. Since the relative exterior measure of M_n in every J is $< 1 - 1/n$, it follows that the interior measure of the set complementary to M_n is greater than $\bar{l}_\sigma/n$ in every $\bar{J}_\sigma$ (of length $\bar{l}_\sigma$); and since no part of the continuum is covered by more than two $\bar{J}$'s, we conclude that the total interior measure of this complementary set in the portion covered by the $\bar{J}_\sigma$ $[\sigma = 1, 2, \cdots \bar{\nu}]$, is $> \frac{1}{2}\cdot(1/n)\cdot\frac{2}{3}l = l/3n$. Therefore the exterior measure of the subset of M_n in I is $< l + \delta - (l/3n)$; and since δ may be chosen arbitrarily small, this exterior measure is $\leqq l - (l/3n)$.

Having thus proved that the relative exterior measure of M_n is $\leqq 1 - (1/3n)$ in every interval, we may now show that the measure of M_n is zero. To this end, we show that *a linear set S whose relative exterior measure is $< 1 - k$, $k > 0$, in every interval is necessarily of measure zero.* For let m = exterior measure of S. For every given positive ϵ we may then enclose S in a set of intervals I_n of total length $< m + \epsilon$. Furthermore, the subset of S in each I_n may be enclosed in a set of intervals of total length $< (1 - k)l_n$, where l_n = length of I_n; and therefore the entire set S, in a set of intervals of total length $< \Sigma_n(1 - k)l_n \leqq (1 - k)(m + \epsilon)$. Therefore

$$(1 - k)(m + \epsilon) > m, \qquad m < \frac{(1 - k)\epsilon}{k},$$

and accordingly $m = 0$. Our theorem is thus proved.

Let Z be the subset of A of zero measure at the points of which the relative exterior measure of A is not 1; and let $H = A - Z$ be the remaining set. Since A and H differ by a set of zero measure, the relative exterior measure of the one is the same at every point as that of the other. Therefore H has the relative measure 1 at every one of its points, and may be thought of as "homogeneous" as to exterior measure. We thus have

COROLLARY 1. *Every linear point set A may be represented as*

$$A = H + Z,$$

where H is a ("homogeneous") set having relative exterior measure 1 *at every one of its points, and Z is of measure zero.*

We obtain a particular case of our theorem if we assume A to be a measurable set. Exterior measure will then be replaced by measure, and relative exterior measure by "relative measure." We thus have

COROLLARY 2. *The relative measure of a measurable set is* 1 *at every one of its points except possibly at those of a set of measure zero.*

Corollary 2 is equivalent to a theorem of Lebesgue-Denjoy.* The present note, therefore, also gives a very simple proof of this important theorem.

So far the author has not succeeded in proving the theorem of this note for higher dimensions, although there seems to be little ground for doubting its validity in n-space.

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A GENERAL FORM OF GREEN'S THEOREM.

BY PROFESSOR P. J. DANIELL.

IN this paper a form of Green's theorem is considered which applies, on the one hand, to the boundary of any set E, measurable Borel, and relates, on the other hand, to potential functions which satisfy a general integral form of Poisson's equation,

$$\int_{B(E)} \frac{\partial V}{\partial n}\, ds = \int_E d\alpha(x, y),$$

where $\alpha(x, y)$ is some function of limited variation in (x, y). In particular it can be used in mathematical physics in problems in which mass (or electric charge) is not distributed continuously.

Let $V_1(x, y)$, $V_2(x, y)$ be two potential functions defined and

* Lebesgue, Leçons sur l'Intégration, pp. 123–124, and Denjoy, loc. cit., pp. 132–137. "Relative measure" is equivalent with Denjoy's "épaisseur." Lebesgue's considerations are indirect (as far as the theorem in question is concerned), being based on properties of integrals. Denjoy's proof is direct, but still comparatively involved and long.