where H is a ("homogeneous") set having relative exterior measure 1 *at every one of its points, and Z is of measure zero.*

We obtain a particular case of our theorem if we assume A to be a measurable set. Exterior measure will then be replaced by measure, and relative exterior measure by "relative measure." We thus have

COROLLARY 2. *The relative measure of a measurable set is* 1 *at every one of its points except possibly at those of a set of measure zero.*

Corollary 2 is equivalent to a theorem of Lebesgue-Denjoy.* The present note, therefore, also gives a very simple proof of this important theorem.

So far the author has not succeeded in proving the theorem of this note for higher dimensions, although there seems to be little ground for doubting its validity in n-space.

UNIVERSITY OF ILLINOIS.

A GENERAL FORM OF GREEN'S THEOREM.

BY PROFESSOR P. J. DANIELL.

IN this paper a form of Green's theorem is considered which applies, on the one hand, to the boundary of any set E, measurable Borel, and relates, on the other hand, to potential functions which satisfy a general integral form of Poisson's equation,

$$\int_{B(E)} \frac{\partial V}{\partial n}\, ds = \int_E d\alpha(x, y),$$

where $\alpha(x, y)$ is some function of limited variation in (x, y). In particular it can be used in mathematical physics in problems in which mass (or electric charge) is not distributed continuously.

Let $V_1(x, y)$, $V_2(x, y)$ be two potential functions defined and

* Lebesgue, Leçons sur l'Intégration, pp. 123–124, and Denjoy, loc. cit., pp. 132–137. "Relative measure" is equivalent with Denjoy's "épaisseur." Lebesgue's considerations are indirect (as far as the theorem in question is concerned), being based on properties of integrals. Denjoy's proof is direct, but still comparatively involved and long.

"differentiable"* in the fundamental square $J(0 \leqq x \leqq 1, 0 \leqq y \leqq 1)$; let

$$\frac{\partial V_1}{\partial x} = v_1, \quad \frac{\partial V_1}{\partial y} = - u_1, \quad \frac{\partial V_2}{\partial x} = v_2, \quad \frac{\partial V_2}{\partial y} = - u_2$$

be summable with their squares in J.

Furthermore assume that u_1, u_2 satisfy R_1; v_1, v_2 satisfy R_2.

R_1. The total variation of $u(x, y)$ varying $y(0 \leqq y \leqq 1)$ is a function of x, finite nearly everywhere and summable in $(0 \leqq x \leqq 1)$.

R_2. The same as R_1, with v in place of u, and with the rôles of x and y interchanged.

In a previous paper† the author has shown that, under these restrictions, we may express

$$\int_{B(E)} \frac{\partial V_1}{\partial n} ds = \int_{B(E)} u_1 dx + v_1 dy$$

in the form

$$\int_E d\alpha_1.$$

In the present paper it is further proved that

$$\int_{B(E)} V_2 \frac{\partial V_1}{\partial n} ds = \int_E V_2 d\alpha_1 + \int_E (u_1 u_2 + v_1 v_2) dxdy.$$

Consider a rectangle r $(a \leqq x \leqq b, c \leqq y \leqq d)$ contained in J. We have

$$\int_{B(r)} u_1 dx = \int_a^b [u_1(x, c) - u_1(x, d)] dx$$

$$= - \int_a^b dx \int_c^d d_y u_1(x, y).$$

In this $\int_c^d d_y u_1(x, y)$ may be regarded as a Stieltjes integral with y as variable, which, by R_1, exists for nearly all values of x and is summable in x. Then

$$- \int_a^b dx \int_c d_y u_1(x, y) = \int_r d\alpha'(x, y)$$

* De la Vallée Poussin, Cours d'Analyse, 3d ed., vol. 1, § 147.

† P. J. Daniell, this Bulletin, Nov., 1918.

will be an absolutely additive function of rectangles r. Similarly

$$\int_{B(r)} v_1 dy = \int_c^d [v_1(b, y) - v_1(a, y)]dy$$

$$= \int_c^d dy \int_a^b d_x v_1(x, y)$$

$$= \int_r d\alpha''(x, y),$$

$$\int_{B(r)} u_1 dx + v_1 dy = \int_r d\alpha(x, y),$$

where $\alpha = \alpha' + \alpha''$.

For nearly all values of x, $(\partial V_2/\partial y) = - u$ is uniformly bounded with respect to y, by R_1. Then, for nearly all values of x, $V_2(x, y)$ is an absolutely continuous function of y and must be of limited variation in y.

By a theorem on integration by parts,*

$$- \int_c^d d_y(V_2 u_1) = - \int_c^d V_2 d_y u_1 - \int_c^d u_1 d_y V_2$$

$$= - \int_c^d V_2 d_y u_1 + \int_c^d u_1 u_2 dy,$$

for nearly all values of x.

Again since V_2 is "differentiable" at every point of J, it is continuous† and uniformly bounded.

If $\max | V_2 | = K$, the total variation of $V_2 u_1$, varying $y (0 \leqq y \leqq 1)$ will be not greater than

$$K \times \text{variation of } u_1 + \int_0^1 | u_1 u_2 | \, dy.$$

u_1, u_2 are summable with their squares in J, so that $u_1 u_2$ is also summable in J, or $\int_0^1 | u_1 u_2 | \, dy$ is a summable function of x where it exists (nearly everywhere). Then $V_2 u_1$ will also satisfy R_1, and similarly $V_2 v_1$ will satisfy R_2.

* P. J. Daniell, *Transactions Amer. Math. Society,* October, 1918.

† De la Vallée Poussin, ibid.

Combining the various facts, we then obtain

$$\int_{B(r)} V_2 u_1 dx = -\int_a^b dx \int_c^d d_y(V_2 u_1)$$

$$= -\int_a^b dx \int_c^d V_2 d_y u_1 + \int_a^b dx \int_c^d u_1 u_2 dy$$

$$= \int_r V_2 d\alpha' + \int_r u_1 u_2 dx dy.$$

The change from repeated to double integrals is legitimate, in the first integral (Stieltjes) because V_2 is uniformly continuous, in the second because $u_1 u_2$ is summable in J. Similarly

$$\int_{B(r)} V_2 v_1 dy = \int_r V_2 d\alpha'' + \int_r v_1 v_2 dx dy.$$

Then

$$\int_{B(r)} V_2(u_1 dx + v_1 dy) = \int_r V_2 d\alpha_1 + \int_r (u_1 u_2 + v_1 v_2) dx dy.$$

This is an equation in which all three expressions are absolutely additive functions of rectangles, and $V_2 u_1$, $V_2 v_1$ satisfy R_1, R_2 respectively; therefore for any set E, measurable Borel, contained in J,

$$\int_{B(E)} V_2 \frac{\partial V_1}{\partial n} ds = \int_E V_2 d\alpha_1 + \int_E (u_1 u_2 + v_1 v_2) dx dy.$$

This was the theorem to be proved and we can rewrite it in the form

$$\int_{B(E)} V_2 \frac{\partial V_1}{\partial n} ds = \int_E V_2 d\alpha_1 + \int_E (\text{grad } V_1 \cdot \text{grad } V_2) dS.$$

Corollary 1. Interchanging V_1, V_2 and subtracting,

$$\int_{B(E)} \left(V_2 \frac{\partial V_1}{\partial n} - V_1 \frac{\partial V_2}{\partial n} \right) ds = \int_E V_2 d\alpha_1 - \int_E V_1 d\alpha_2.$$

Corollary 2. Instead of this make $V_1 = V_2 = V$.

$$\int_{B(E)} V \frac{\partial V}{\partial n} ds = \int_E V d\alpha + \int_E \text{grad}^2\, V dS.$$

Three dimensions. The case of three dimensions is more valuable in applications, and the proof is exactly similar. We content ourselves with the mere statement. [Our previous notation is altered; u_1 takes the place of v_1, and v_1 of $-u_1$.] Let V_1, V_2 be two potential functions, defined and "differentiable" in the cube J $(0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1,\ 0 \leqq z \leqq 1)$; let the six partial derivatives be summable with their squares in J; and let $\partial V_1/\partial x$, $\partial V_2/\partial x$ satisfy R_1; $\partial V_1/\partial y$, $\partial V_2/\partial y$ satisfy R_2; $\partial V_1/\partial z$, $\partial V_2/\partial z$ satisfy R_3.

R_1. The total variation of u varying $x (0 \leqq x \leqq 1)$ is a function of (y, z) finite nearly everywhere and summable in $(0 \leqq y \leqq 1,\ 0 \leqq z \leqq 1)$.

R_2, R_3. The same as R_1 with v, w in place of u and with cyclical interchanges of x, y, z.

If the element of normal is drawn outwards,

$$\int_{B(r)} \frac{\partial V_1}{\partial n} dS = \int_r d\alpha_1$$

is an absolutely additive function of rectangular parallelopipeds r and $\alpha_1(x, y, z)$ is a function of limited variation in J. Then we can define for any set E, in J, measurable Borel,

$$\int_{B(E)} \frac{\partial V_1}{\partial n} dS = \int_E d\alpha_1$$

and Green's theorem becomes

$$\int_{B(E)} V_2 \frac{\partial V_1}{\partial n} dS = \int_E V_2 d\alpha_1 + \int_E (\text{grad } V_1 \cdot \text{grad } V_2) d \text{ vol.}$$

The two corresponding corollaries follow immediately.

RICE INSTITUTE,
HOUSTON, TEXAS.