THE SELF-DUAL PLANE RATIONAL QUINTIC.

BY PROFESSOR L. E. WEAR.

A SELF-DUAL curve is defined to be a curve which has the same number of cusps and double points as it has inflexional tangents and double tangents respectively; and furthermore there are correlations—including polarities—which send the curve into itself.

Haskell, in this BULLETIN, January, 1917, found the maximum number of cusps of an algebraic plane curve, and enumerated the self-dual curves. The well known binomial curves

$$x_1^n = x_0^{n-r} x_2^r$$

have been extensively studied and shown to be self-dual.* The case of the rational plane quartic has been considered in my dissertation at the Johns Hopkins University.†

We here consider briefly the quintic. Since the class of the curve is to equal the order, we have as the fundamental equation,

$$n = n(n-1) - 2d - 3c,$$

where d is the number of double points and c the number of cusps. Hence we have for the quintic,

$$2d + 3c = 15,$$

an equation which has three solutions, as follows:

$$\begin{aligned} &(1)\quad d = 0, \qquad c = 5, \\ &(2)\quad d = 3, \qquad c = 3, \\ &(3)\quad d = 6, \qquad c = 1. \end{aligned}$$

Case (3) may arise from the degenerate quintic composed of a conic and a cuspidal cubic.

The second case, that of the rational quintic, is the one to be considered here. Furthermore, we consider the curve which

* Loria, Spezielle Ebene Kurven, p. 308; Wieleitner, Algebraische Kurven, p. 136; Snyder, *American Journal*, vol. 30; Winger, *American Journal*, vol. 36.

† "The self-dual plane rational quartic," Dissertation, Johns Hopkins University, May, 1913.

is self-dual in all possible ways, which will be invariant under the largest possible group of transformations. The cusps then will be distinct. By taking the products of the correlations two at a time we obtain the collineations of a collineation group under which the curve is self-projective. These must interchange cusps, say, in all possible ways, and hence the curve must be invariant under a G_6 composed of a cyclic g_3, the elements of which interchange the cusps cyclically, and three elements, obtained by adding to the g_3 an element of period two, and which leave one cusp fixed while interchanging the other two.

Now the equations of the rational quintic invariant under the dihedral G_6 are*

$$(1) \qquad x_0 = t^5 + 5t^2, \qquad x_1 = 5t^3 + 1, \qquad x_2 = t^4 + t.$$

The flexes are $t^3 + 1$, and the cusps $t^3 - 1$. The G_6 is generated by the elements

$$(2) \qquad t' = \omega t, \qquad t' = 1/t, \qquad (\omega^3 = 1),$$

with the appropriate ternary transformations

$$(3) \qquad x_0' = x_0, \qquad x_1' = \omega x_1, \qquad x_2' = \omega^2 x_2.$$

$$(4) \qquad x_0' = x_1, \qquad x_1' = x_0, \qquad x_2' = x_2.$$

Let us now add a correlation which will send any point of the curve into a line of the curve, and vice versa. In particular we desire a correlation that will interchange cusps and flexes, and likewise double points and double lines.

In order to obtain the correlation we need the line equations of the curve, which are obtained by taking the Jacobians of (1) two at a time, and are

$$(5) \quad \xi_0 = 5\tau^3 - 1, \quad \xi_1 = \tau^2(\tau^3 - 5), \quad \xi_2 = -10\tau(\tau^3 - 1).$$

The binary transformation $t\tau = -1$ will send the cusp $t = 1$ into the flex line $\tau = -1$, and conversely. Let us ask that this send *any point* of the curve into a *line* of the curve. Now any point of the curve is

$$(6) \qquad \xi_0(t^5 + 5t^2) + \xi_1(5t^3 + 1) + \xi_2(t^4 + t) = 0$$

and any line is,

$$(7) \qquad x_0(5\tau^3 - 1) + x_1(\tau^3 - 5) - x_2 10\tau(\tau^3 - 1) = 0.$$

* See Winger, l. c., p. 73.

Making the substitution $t = -1/\tau$ in (6), we have, after simplifying,

$$\xi_0(5\tau^3 - 1) + \xi_1(\tau^5 - 5\tau^2) + \xi_2(\tau - \tau^4) = 0. \tag{8}$$

By identifying (7) with (8) there results the polarity

$$\xi_0 = x_0, \qquad \xi_1 = x_1, \qquad \xi_2 = 10x_2. \tag{9}$$

Combining (9) with the elements of the collineation group G_6, we obtain altogether six correlations which leave the curve unaltered. The latter statement is true since elements of the G_6 send any point of the curve into a second point, and this transformation followed by (9) must send the original point into a line of the curve. The correlations, with the elements of the G_6, make up a G_{12} of collineations and correlations under which the curve is invariant. The following table gives the elements of the group, binary and ternary:

Collineations.

	x_0'	x_1'	x_2'	t'
1:	$= x_0,$	$= x_1,$	$= x_2,$	$= t.$
S:	$x_0,$	$\omega x_1,$	$\omega^2 x_2,$	$\omega t.$
S^2:	$x_0,$	$\omega^2 x_1,$	$\omega x_2,$	$\omega^2 t.$
T:	$x_1,$	$x_0,$	$x_2,$	$1/t.$
ST:	$x_1,$	$\omega x_0,$	$\omega^2 x_2,$	$\omega/t.$
S^2T:	$x_1,$	$\omega^2 x_0,$	$\omega x_2,$	$\omega^2/t.$

Correlations.

	ξ_0	ξ_1	ξ_2	τ
Π_0:	$= x_0,$	$= x_1,$	$= 10x_2,$	$= -1/t.$
$\Pi_0 S$:	$x_0,$	$\omega x_1,$	$10\omega^2 x_2,$	$-\omega/t.$
$\Pi_0 S^2$:	$x_0,$	$\omega^2 x_1,$	$10\omega x_2,$	$-\omega^2/t.$
$\Pi_0 T$:	$x_1,$	$x_0,$	$10x_2,$	$-t.$
$\Pi_0 \cdot (ST)$:	$x_1,$	$\omega x_0,$	$10\omega^2 x_2,$	$-\omega^2 t.$
$\Pi_0 \cdot (S^2T)$:	$x_1,$	$\omega^2 x_0,$	$10\omega x_2,$	$-\omega t.$

It is easily verifiable that these elements have the group properties. Only the first four of the correlations are polarities, and of these $\Pi_0 T$ alone refers to a real conic, the equation of which is

$$x_0 x_1 + 5x_2^2 = 0.$$

This conic is tangent to the curve at $t = 0$, $t = \infty$, and intersects the curve at six other points. At one of the latter points a tangent to the conic is tangent to the curve at some other point. We may summarize with this theorem: *The self-dual plane rational quintic admitting of the greatest possible number of correlations is invariant under a G_{12} consisting of collineations and correlations.*

THROOP COLLEGE,
February, 1919.

GROUPS CONTAINING A RELATIVELY LARGE NUMBER OF OPERATORS OF ORDER TWO.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society March 29, 1919.)

§ 1. *Introduction.*

IT is well known that every group which contains at least one operator of order 2 must contain an odd number of such operators and that there is an infinite number of groups such that each of them contains exactly $2m + 1$ operators of order 2, where m is an arbitrary positive integer or 0. It is also known that if exactly one half of the operators of a group are of order 2 then the order of this group must be of the form $2(2m + 1)$ and it must be the dihedral or the generalized dihedral group of this order. Moreover, it has been proved that a group G of order

$$g = 2^\alpha(2m + 1)$$

cannot contain more than $2^\alpha m + 2^\alpha - 1$ operators of order 2, α being an arbitrary positive integer, and whenever G contains this number of operators of order 2 it is either the abelian group of order 2^α and of type $(1, 1, 1, \cdots)$ or it is the direct product of the abelian group of order $2^{\alpha-1}$ and of type $(1, 1, 1, \cdots)$ and the dihedral or the generalized dihedral group of order $2(2m + 1)$.*

* G. A. Miller, this BULLETIN, vol. 13 (1907), p. 235.