ON THE RECTIFIABILITY OF A TWISTED CUBIC.

BY PROFESSOR TSURUICHI HAYASHI.

UNDER the same title in the November number of this BULLETIN, 1918, volume 25, page 87, Dr. Mary F. Curtis showed that, given the twisted cubic

$$x_1 = at, \qquad x_2 = bt^2, \qquad x_3 = ct^3, \qquad abc \neq 0,$$

the condition that it be a helix is precisely the condition that it be algebraically rectifiable. Since her proof is an application of the common differential geometry, the coordinates x_1, x_2, x_3 are rectangular. The parametrical equations of the most general twisted cubics in rectangular coordinates x, y, z, are

$$x = f_1(t)/F(t), \qquad y = f_2(t)/F(t), \qquad z = f_3(t)/F(t),$$

where F, f_1, f_2, f_3 are polynomials of degree 3 in the parameter t. By increasing t by a constant, and taking the axes along the tangent, the principal normal, and the binormal at the point on the curve where t is infinite, these equations are reduced to

$$(1) \quad x = \frac{a_0t^2 + a_1t + a_2}{t^3 + dt + e}, \quad y = \frac{b_1t + b_2}{t^3 + dt + e}, \quad z = \frac{c_2}{t^3 + dt + e},$$

as Mr. W. H. Salmon has done in treating the twisted cubics of constant torsion.* But I will consider here the algebraic rectifiability of a less general type of twisted cubics whose equations are

$$x = a_1t^3 + a_2t^2 + a_3t + a_4,$$
$$y = b_1t^3 + b_2t^2 + b_3t + b_4,$$
$$z = c_1t^3 + c_2t^2 + c_3t + c_4.$$

This type is even more general than that treated by Dr. Curtis, and contains the cubics which have the plane at infinity as their osculating plane.

By changing the value of t by a constant, transferring the origin, and taking the axes of coordinates along the tangent, the principal normal and the binormal at the point on the

* *Messenger of Mathematics*, vol. 45 (1915–1916), pp. 125–128.

curve where t is then equal to zero, the equations may be written

$$(2) \qquad x = a_1t^3 + a_3t, \qquad y = b_1t^3 + b_2t^2, \qquad z = c_1t^3.$$

From these equations we get

$$\Sigma\dot{x}^2 = (3a_1t^2 + a_3)^2 + (3b_1t^2 + 2b_2t)^2 + (3c_1t)^2,$$

$$\Sigma(\dot{y}\ddot{z} - \dot{z}\ddot{y})^2 = (6b_2c_1t^2)^2 + (6a_3c_1t)^2 + (-6a_1b_2t^2 + 6a_3b_1t + 2a_3b_2)^2$$

and

$$\dot{x}\ddot{y}\dddot{z} = 12a_3b_2c_1.$$

The ratio of the radii of torsion and curvature

$$\frac{T}{R} = -\left(\frac{\Sigma(\dot{y}\ddot{z} - \dot{z}\ddot{y})^2}{\Sigma\dot{x}^2}\right)^{3/2} \cdot \frac{1}{|\dot{x}\ddot{y}\dddot{z}|}$$

can now be expressed in terms of t. Equating this ratio to a constant, we get only two conditions among the five coefficients a_1, a_3; b_1, b_2; c_1, which are necessary and sufficient in order that the twisted cubic be a helix:

$$(3) \qquad b_1 = 0 \qquad \text{and} \qquad 9a_3^2c_1^2 = 12a_1a_3b_2^2 + 4b_2^4.$$

Hence the theorem:—*In order that the twisted cubic of the type* (2) *be a helix it is necessary and sufficient that conditions* (3) *be satisfied.* The value of T/R for the helix is equal to $\pm 2b_2^2/3a_3c_1$.

In this case,

$$\Sigma\dot{x}^2 = (9a_1^2 + 9c_1^2)t^4 + (6a_1a_3 + 4b_2^2)t^2 + a_3^2,$$

so that the arc of the twisted cubic

$$s = \int_{t_0}^{t_1} (\Sigma\dot{x}^2)^{1/2}dt$$

is algebraic when and only when conditions (3) hold. Hence the theorem: *The twisted cubic of the type* (2) *is algebraically rectifiable when and only when it is a helix,* a generalization of Dr. Curtis's theorem.

Lastly I will add here a theorem on twisted cubics of constant torsion and of constant curvature. It can be shown that for the twisted cubic of the type (2) of constant torsion it is necessary and sufficient that

$$(4) \qquad b_1 = 0, \qquad a_1^2 + c_1^2 = 0, \qquad 3a_3c_1^2 = 2b_1b_2^2.$$

But this is comprised in Mr. Salmon's theorem in his paper referred to. It can be also shown that for the twisted cubic of the type (2) of constant curvature, the same conditions (4) are necessary and sufficient. This was carried out by Mr. S. Narumi, one of our students. Hence the theorem: *The twisted cubics of the type* (2) *of constant torsion are of constant curvature, and conversely those of constant curvature are of constant torsion; they are all imaginary and satisfy the conditions* (4). A similar problem on the twisted cubics of the most general type (1), that is rather more interesting, as Prof. T. Kubota, one of our colleagues, says, remains unsolved.

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SOME GENERALIZATIONS OF THE SATELLITE THEORY.

BY PROFESSOR R. M. WINGER.

§1. *The Rational Cubic.*

In a forthcoming paper* in the *American Journal of Mathematics* the author considers the satellite line of the cubic and some related curves. The object of the present note is to point out how the principal results there obtained can be generalized.

For the rational cubic R^3,

$$x_1 = 3t^2, \quad x_2 = 3t, \quad x_3 = t^3 + 1,$$

the relation connecting a point τ and its tangential t is

$$t\tau^2 + 1 = 0.$$

And the condition that $3n$ points lie on a curve C_n is

$$s_{3n} = (-1)^n,$$

where s refers to the product of the t's†. From these equations we have at once the following theorems:

1. *The tangentials of the points in which a C_n' cuts R^3 lie on a second C_n.*

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* "On the satellite line of the cubic," read before the San Francisco section of the American Mathematical Society, April 6, 1918.

† Winger, "Involutions on the rational cubic," this BULLETIN, Oct., 1918.