A SET OF COMPLETELY INDEPENDENT POSTULATES FOR THE LINEAR ORDER η*.

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Professor E. V. Huntington has published† three sets of completely independent postulates for serial order. His set $A$ involves four postulates, which is as high a number of postulates as had been proved completely independent. In the present paper are given seven postulates which form a categorical and completely independent set for the linear order.

Our basis is a class of elements $[p]$ and an undefined dyadic relation (called ‘less than’) among the elements. If we are given two elements $p_1, p_2$ and if the relation $p_1$ less than $p_2$ holds, we will symbolize it by $p_1 < p_2$. If the relation $p_1$ less than $p_2$ does not hold, we will symbolize it by $p_1 < p_2$.

Our postulates are:

I. If $p_1 < p_2$, then $p_2 < p_1$.
II. If $p_1 < p_2$, then $p_2 < p_1$; $p_1, p_2$ distinct.
III. If $p_1 < p_2$ and $p_2 < p_3$, then $p_1 < p_3$.
IV. If $p_1 < p_2$, then there exists a $p_3$ such that $p_1 < p_3$ and $p_3 < p_2$.
V. For every $p_1$ there exists a $p_2$ such that $p_2 < p_1$.
VI. For every $p_1$ there exists a $p_2$ such that $p_1 < p_2$.
VII. The class of elements $[p]$ form a denumerable set.

That the set is categorical follows from the fact that the seven postulates stated are the necessary and sufficient conditions for the linear order $\eta$. To show complete independence it will be necessary to cite 128 $(2^{7})$ examples showing all possible combinations ($\pm \pm \pm \pm \pm \pm \pm \pm$) of our postulates holding and not holding. This is done by giving eight definitions of $<$, and sixteen sets of points such that each definition is applicable to every one of the sets, and every combination

* The linear order $\eta$ is an ordered set equivalent to that of all the rational numbers.
† “Sets of completely independent postulates for serial order.” This Bulletin, March, 1917. This paper contains a bibliography of complete independence.
of definition of $<$ and set yields a different example. The eight definitions give the eight $(\pm \pm \pm \pm)$ groups of cases for the implicational postulates I, II and III, whereas each of the sixteen sets gives all the eight cases where any particular set $(\pm \pm \pm \pm)$ of the existential postulates IV, V, VI and VII hold or do not hold.

For the independence examples, the set $[p]$ consists of points on a line such that

$$
1) - - - - \quad p = -3, \quad -2 \leq p \leq 2, \quad p = 3 \text{ and } p \text{ real.}
$$

$$
2) - - - + \quad p = -3, \quad -2 \leq p \leq 2, \quad p = 3 \text{ and } p \text{ rational.}
$$

$$
3) - - + - \quad p = -3, \quad -2 \leq p \leq 3, \quad p \text{ real.}
$$

$$
4) - + + - \quad p = -3, \quad -2 \leq p < 3, \quad p \text{ rational.}
$$

$$
5) + - - + \quad -3 < p \leq 3, \quad p = 3 \text{ and } p \text{ real.}
$$

$$
6) + - + - \quad -3 < p \leq 3, \quad p = 3 \text{ and } p \text{ rational.}
$$

$$
7) + + - + \quad -3 < p \leq -3/2, \quad 2 \leq p \leq 3, \quad p \text{ real.}
$$

$$
8) + + + - \quad -3 < p \leq -3/2, \quad 2 < p < 3, \quad p \text{ rational.}
$$

$$
9) + + + + \quad -3 < p < 3, \quad p \text{ real.}
$$

$$
10) + + + + \quad -3 < p < 3, \quad p \text{ rational.}
$$

$$
11) + + + - \quad -3 \leq p < 3, \quad p \text{ real.}
$$

$$
12) + + + + \quad -3 \leq p < 3, \quad p \text{ rational.}
$$

$$
13) + + + + \quad -3 \leq p < 3, \quad p \text{ real.}
$$

$$
14) + + + + \quad -3 < p < 3, \quad p \text{ rational.}
$$

$$
15) + + + + \quad -3 < p < 3, \quad p \text{ real.}
$$

$$
16) + + + + \quad -3 < p < 3, \quad p \text{ rational.}
$$

A definition of $<$ requires that whenever we are given two numbers of our set $p_1p_2$ we have a criterion whereby we can tell whether the relation $p_1 < p_2$ holds or does not hold. In all the eight definitions of $<$ the relation holds for any pair of numbers $p_1p_2$ if it holds in the case of ordinary linear order,

$$1') - - - - \quad \text{except } 0 < 1, \quad -1 < -2, \quad 0 < -1 \text{ and } 0 < -2.$$

$$2') - - - + \quad \text{except } 1 < -1, \quad 1 < 0, \quad 0 < -1, \quad p_1 < -1, \quad p_1 < 0, \quad p_1 < 1, \quad -1 < p_2, \quad 0 < p_2 \text{ and } 1 < p_2; \quad p_1 + - 1, \quad 0, \quad 1; \quad p_2 + - 1, \quad 0, \quad 1.$$

$$3') - + - + \quad \text{except } 0 < -m/2^n, \quad n \text{ positive integer and } m \text{ odd positive integer.}
$$

$$4') - + + + \quad \text{except } p_1 < -1, \quad p_1 < 0, \quad p_1 < 1, \quad -1 < p_2, \quad 0 < p_2, \text{ and } 1 < p_2; \quad p_1 + - 3, \quad p_2 + - 1, \quad 0, \quad 1.$$

$$5') + - - - \quad \text{and } p_3 - p_1 < 1/3.$$

$$6') + + + - \quad \text{and } p_3 - p_1 = m/2^n, \quad n \text{ positive integer and } m \text{ odd integer.}
$$

$$7') + + + + \quad \text{except } 0 < -m/2^n \text{ and } -m/2^n < 0, \quad n \text{ positive integer and } m \text{ odd positive integer.}
$$

$$8') + + + + \quad \text{with no exceptions.}
$$

To illustrate: The independence example where postulates II, III, V, and VII hold and postulates I, IV and VI do not hold $(-+-+-+-)$ is definition 4' used on set 6.

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