which is seen to agree with (18), except for the fact that the coefficient of $\tau - \tau_0$ in the exponent of e in (20) is the reciprocal of that in (19). This brings out the relation between systems (19) and (17). This is readily generalized to all systems like (17) which lead to self-adjoint differential equations of the second order with boundary conditions.

HARVARD UNIVERSITY,
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ON A PENCIL OF NODAL CUBICS.

BY PROFESSOR NATHAN ALTSHILLER-COURT.

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CONSIDER a pencil Γ of nodal cubics having in common three collinear points, the double point, and the two tangents at this point.

1. Let Γ_n be one of the cubics of the pencil Γ. A variable secant passing through one of the three basic collinear points A, B, C, say A, cuts Γ_n in pairs of points which are projected from the double point O by an involution of rays, the two tangents OT_1, OT_2 (T_1, T_2 are points of the basic line ABC) to Γ_n at O being a pair of conjugate elements in this involution.[1] The lines OB, OC, are obviously another pair of conjugate elements in this involution. The double lines OM_n, OM_n', of this involution project from O the two points of contact M_n, M_n' of the two tangents from A to Γ_n.

When Γ_n describes the pencil Γ the two pairs of lines OT_1, OT_2 and OB, OC, remain fixed, by hypothesis, hence the involution

$$\text{(I)} \qquad O(T_1T_2,\ BC,\ M_nM_n,\ M_n'M_n')$$

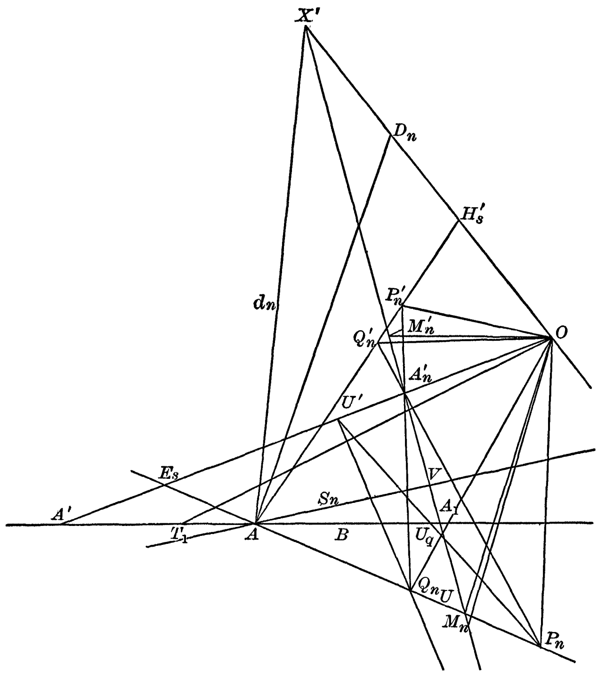
is fixed, and therefore also its doubles lines OM_n, OM_n'. Consequently: *The points of contact of the pairs of tangents from one of the basic points to curves of the pencil* Γ *lie on two fixed lines passing through the double point.*

2. The corresponding point A_n' of A on Γ_n is projected

[1] M. Chasles, "Mémoire sur la construction des racines des équations du troisième et du quatrième degré," *Comptes Rendus de l'Académie des Sciences*, tome 41, p. 677. E. de Jonquières, Mélanges de Géométrie pure, 1856, p. 180.

from O by a line OA_n' which is the harmonic conjugate of OA with respect to OT_1, OT_2.[2]

When Γ_n varies the three lines OA, OT_1, OT_2 remain fixed, by hypothesis, therefore: *The corresponding points on the*



cubics of the pencil Γ *of any one of the basic points lie on a straight line passing through the double point.*

3. The chord of contact M_nM_n' of the two tangents AM_n, AM_n' from A to Γ_n (1) passes through the corresponding point A_n' (2) of A on Γ_n.[3] The two lines OA_n', $A_n'M_nM_n'$ meet the line ABC in two points A', A_1 harmonically separated by B, C, according to a theorem of McLaurin.[4]

When Γ_n varies the point A' remains fixed (2), and since B, C are also fixed, hence A_1 is a fixed point. Thus: *The chords of contact of the pairs of tangents from one of the basic*

[2] See, for instance, W. Binder, Theorie der unicursalen Plancurven, p. 293.

[3] de Jonquières, loc. cit., p. 226, prop. VII.

[4] de Jonquières, loc. cit., p. 237, prop. XIV.

points to the cubics of the pencil Γ *pass through a fixed point of the basic line.*

4. Let D_n be the tangential of A on Γ_n, i.e., the point where Γ_n is met again by the tangent to this cubic at the point A. In the involution (I) (1) to the line OD_n corresponds the line OA. Since neither this involution, nor the line OA vary with Γ_n, the line OD_n is therefore fixed. Consequently: *The tangentials of any of the basic points on the cubics of the pencil* Γ *lie on a straight line passing through the double point.*

5. Let P_n be any point on Γ_n. The line AP_n meets Γ_n again in Q_n, and the two lines OP_n, OQ_n are conjugate in the involution of rays (I) (1).

When P_n varies with Γ_n on the line OP_n, the line OQ_n, being the conjugate of the fixed line OP_n in (I), will remain fixed. Hence: *The lines projecting from a basic point the range of points determined by the pencil* Γ *on any line passing through the double point, meet the respective cubics again on a fixed straight line passing through the double point.*

5*a*. The corresponding points of A describe a straight line (2), hence (5): *The lines joining a basic point to its corresponding points on the cubics of the pencil* Γ *meet the respective curves again on a straight line passing through the double point.*

6. The pairs of points determined on Γ_n by a variable secant passing through P_n are projected from O by an involution of rays in which OT_1, OT_2 are a couple of conjugate elements, and the lines OA, OQ_n (5) are another.[1] In this involution to the ray OP_n corresponds the line OL_n projecting from O the tangential L_n of P_n on Γ_n.[5]

If the line OP_n is maintained fixed while Γ_n describes the pencil Γ, the point Q_n will describe the range of points OQ_n (5). Thus when Γ_n varies the involution of rays just considered has besides the fixed couple OT_1, OT_2 also the fixed couple OA, OQ_n, hence this involution is fixed, and the conjugate OL_n in this involution of the fixed ray OP_n is also fixed. Consequently: *The tangentials of the range of points determined by the curves of the pencil* Γ *on a line passing through the double point, lie on a straight line also passing through the double point.*

6*a*. Since L_n describes a line passing through O, its tangentials, which are the second tangentials of P_n, will also describe a straight line. In turn we may consider the tan-

[5] Binder, loc. cit.

gentials of these tangentials, etc. Thus: *The tangentials of any given order of the range of points determined by the pencil* Γ *on a line passing through the double point, lie on a straight line also passing through the double point.*

6*b*. We have seen that the tangentials of A describe a straight line (4), hence (6*a*): *The tangentials of any given order of a basic point on the cubics of the pencil* Γ *lie on a straight line passing through the double point.*

7. The double elements OT_n, OT_n' of the involution of rays $O(T_1T_2, AQ_n)$ (6) project from O the points of contact T_n, T_n' of the tangents P_nT_n, P_nT_n' from P_n to Γ_n. Since this involution remains fixed when P_n varies with Γ on the line OP_n (6), we have: *The points of contact of the pairs of tangents drawn to the cubics of the pencil* Γ *from the points which these curves respectively determine on a line passing through the double point, lie on two straight lines passing through the double point.*

8. The lines OA_n', $A_n'A_1$ (3) meet the secant AP_nQ_n (6) in two points harmonically separated by P_n and Q_n.[4] Let $U \equiv (AP_nQ_n, A_1A_n')$.

Let R_n be any other point of Γ_n, and S_n the third point of intersection of Γ_n with AR_n. The lines OA_n', $A_n'A_1$ (3) determine on the secant AR_nS_n two points harmonically separated by R_n, S_n.[4] Let $V \equiv (R_nS_n, A_1A_n')$.

If the lines OP_n, OR_n are kept fixed when Γ_n describes the pencil Γ, the points P_n, R_n will describe two ranges of points. The points Q_n, S_n will describe two other ranges of points on the lines OQ_n, OS_n (5). Hence the point U will also describe a straight line, namely the harmonic conjugate of the fixed line OA_n' (2) with respect to the couple of lines OP_n, OQ_n. Similarly V will describe the harmonic conjugate of OA_n' with respect to OR_n, OS_n. Thus we have

$$\begin{aligned}(P_n \cdots) \barwedge A(P_n \cdots) \barwedge A_1(U \cdots) \\ \barwedge (V \cdots) \barwedge A(V \cdots) \barwedge (R_n \cdots).\end{aligned}$$

It follows immediately from this construction that (*i*) when P_n coincides with O, the point R_n likewise coincides with O, and (*ii*) when P_n coincides with the point (OP_n, ABC), the point R_n coincides with (OR_n, ABC). Hence: *The lines joining the pairs of points determined by a variable cubic of the pencil* Γ *on two fixed lines passing through the double point pass through a fixed point of the basic line.*

9. The tangent P_nL_n to Γ_n at P_n joins two points which Γ_n determines on the two lines OP_n (5), OL_n (6). Consequently (8): *The tangents to the curves of the pencil* Γ *at their points of intersection with a line passing through the double point, pass through a fixed point of the basic line.*

9*a*. This proposition (9) may be applied to the range of points A_n' (2) and also to the range of points D_n (4). The reader may formulate the resulting theorems.

10. The tangents P_nT_n, P_nT_n' from P_n to Γ_n have their points of contact T_n, T_n' on two fixed straight lines (7). Each of these two tangents passes through a fixed point of the basic line, when Γ_n varies (9), hence we have, relatively to the variable point P_n of the range OP_n: *The pairs of tangents drawn to the cubics of the pencil* Γ *from the points which these curves respectively determine on a fixed line passing through the double point, pass through a pair of fixed points of the basic line.*

11. The corresponding point P_n' of P_n on Γ_n lies on the line OP_n' which is harmonically separated from OP_n by the tangents OT_1, OT_2 to Γ_n at O,[2] therefore when P_n varies with Γ_n on OP_n, the line OP_n' remains fixed. Thus: *A line through the double point meets the cubics of the pencil* Γ *in a range of points the corresponding points of which on the respective curves lie on a straight line passing through the double point*; and by virtue of (8): *The lines joining the pairs of corresponding points thus obtained pass through a fixed point of the basic line.*

12. Since the points of contact T_n, T_n' of the tangents from the variable point P_n of OP_n to the variable cubic Γ_n describe two straight lines (7), hence the variable line T_nT_n' meets ABC in a fixed point (8). Thus: *The chords of contact of the pairs of tangents drawn to the cubics of the pencil* Γ *from the points which these curves respectively determine on a fixed line passing through the double point, meet the basic line in a fixed point.*

13. The chord of contact of the two tangents from A_n' (2) to Γ_n passes through the conjugate A of A_n' on Γ_n[3] and meets the line $A_n'M_nM_n'$ (3) in the harmonic conjugate X' of the point (OA, M_nM_n') with respect to the couple M_n, M_n'.[4]

The pairs of lines projecting from A the couples of corresponding points on Γ_n form an involution of rays, one of the double elements of which is the line AO joining A to the double point O. Since M_n, M_n' are a pair of corresponding points (3), the line AX' is the second double element of this involution of rays.

The line AX' has thus a definite geometric meaning relative to the point A on Γ_n. It is sometimes more convenient to speak of this line in connection with the point A alone, without having to mention the corresponding point of A. Such a line as AX' shall be referred to as "the double line" through A relative to Γ_n.

When A_n' varies with Γ_n (2) the harmonic pencil $O(M_nM_n'AX')$ has three rays OM_n, OM_n' (1), OA, fixed, hence the fourth ray OX' is also fixed. Consequently: *The double line, relative to a curve of the pencil* Γ, *through a basic point meets the chord of contact of the two tangents to this cubic from the same basic point, on a fixed line passing through the double point.*

It may readily be shown that the line OX' is identical with the locus OD_n of the tangentials of A(4).

14. Let K be the point of intersection of the line P_nR_n(8) with ABC, and W_n the third point of intersection of P_nR_n with Γ_n. Let P_s, R_s, W_s be the points of intersection of the lines OP_n, OR_n, OW_n with any other cubic, say Γ_s, of the pencil Γ. The lines P_nR_n, P_sR_s intersect in K, by virtue of prop. (8), i.e., P_s, R_s, K are collinear.

Again, the lines P_nW_n, P_sW_s, intersect on ABC, by virtue of the same prop. (8). But $P_nW_n \equiv P_nR_n$ by construction, hence P_sW_s passes through K, i.e., P_s, W_s, K are collinear, consequently P_s, R_s, W_s are collinear. Thus: (*a*) *If three lines passing through the double point meet one of the cubics of the pencil* Γ *in three collinear points, they meet every cubic of the pencil in three collinear points.* (*b*) *The bases of these triads of points concur on the basic line of the pencil.*

15. The lines P_nA, P_nA_n' projecting from the point P_n (5) the couple of corresponding points A, A_n' (2) on Γ_n meet Γ_n again in a pair of corresponding points.[6] The line P_nA meets Γ_n again in Q_n (6), hence P_nA_n' meets Γ_n again in the corresponding point Q_n' of Q_n on Γ_n. Thus the corresponding points P_n', Q_n' of P_n, Q_n on Γ_n are collinear with A, and therefore the lines OP_n', OQ_n', are conjugate in the involution (I) (1).

The lines AP_n, AP_n', projecting from A the pair of corresponding points P_n, P_n' are harmonically separated by the double lines AO, AX' of the involution projecting from A the pairs of corresponding points on Γ_n (13). The line OX' cuts

[6] de Jonquières, loc. cit., p. 239, prop. XV.

the harmonic pencil $A(OX'P_nQ_n)$ in four harmonic points O, X', H_s, H_s'. The corresponding points P_n', Q_n' of the points P_n, Q_n are thus the intersection of AH_s' with the harmonic conjugates OP_n', OQ_n' of OP_n, OQ_n with respect to OT_1, OT_2.

If the secant $s \equiv AP_nQ_n$ is maintained fixed, while Γ_n varies, the pair of lines OP_n, OQ_n will describe the involution (I) of which the pair of points P_n, Q_n will describe the section by s. The lines OP_n', OQ_n' will describe the same involution (I). The line OX' will remain fixed (13), the points X' and $U(8)$ lying on the line $A_1A_n'M_nM_n'$ (2, 3). Of the four harmonic points O, X', H_s, H_s' the points O, H_s are fixed, hence the two others H_s', X' will describe on the fixed line OX' two superposed projective ranges the double elements of which are the points O and H_s. Thus:

$$O(P_n'Q_n';\ \cdots) \barwedge O(P_nQ_n;\ \cdots) \doublebarwedge (P_nQ_n;\ \cdots) \barwedge (U\ \cdots)$$
$$\doublebarwedge A_1(U\ \cdots) \doublebarwedge (X'\ \cdots) \barwedge (H_s'\ \cdots) \doublebarwedge A(H_s'\ \cdots).$$

This construction establishes a one-to-two projective correspondence between the pencil of lines $A(H_s'\ \cdots)$ and the pairs of lines of the involution (I). Consequently[7]: *The corresponding points, on the cubics of the pencil Γ, of the points determined by this pencil on a fixed line passing through a basic point, lie on a nodal cubic having the same double point as the given pencil and passing through the basic point considered.*

When the lines OP_n, OQ_n coincide with the couple OB, OC of (I) the above construction shows that the harmonic conjugates of OB, OC with respect to the couple OT_1, OT_2 are the two tangents to this cubic at its double point O.

16. The lines OP_n, OQ_n (8) being harmonically separated by the double elements OM_n, OM_n' of (I), the points $U_p \equiv (OP_n,\ M_nM_n')$, $U_q \equiv (OQ_n,\ M_nM_n')$ are harmonically separated by M_n, M_n'. On the other hand M_n, M_n' being a couple of corresponding points on Γ_n (1), the lines P_nM_n, P_nM_n' are conjugate in the involution of lines projecting from P_n the couples of corresponding points on Γ_n, and since one of the double elements of this involution of rays is P_nO, therefore the second double element passes necessarily through the point U_q. Thus P_nU_q is the double line (13) of P_n relative to Γ_n. Similarly the line Q_nU_p is the double line relative to Γ_n through Q_n.

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[7] de Jonquières, loc. cit., pp. 170–171.

The four points P_n, Q_n, U_p, U_q, determine a complete quadrangle the diagonal points of which are $O \equiv (P_nU_p, Q_nU_q)$, $U \equiv (P_nQ_n,\ U_pU_q \equiv A_1A_n')$, $U' \equiv (P_nU_q,\ Q_nU_p)$. Thus OU' is the harmonic conjugate of OU with respect to OP_n, OQ_n, hence OU' is identical with OA' (8), i.e., U' lies on the fixed line OA'. Let $E_s \equiv (OA', P_nQ_n)$. The harmonic pencil $U(E_sA_n'OU')$ determines on OA' the four harmonic points E_s, A_n', O, U'.

If the secant $s \equiv AP_nQ_n$ is kept fixed, while the cubic Γ_n varies, the pair of lines OP_n, OQ_n will describe the involution (I) of which the couple of points P_n, Q_n describe the section by the fixed line s. Of the four harmonic points E_s, A_n', O, U' the points O, E_s remain fixed, hence A_n', U' describe on the fixed line OA' two superposed projective ranges with the points O and E_s as the double elements of the projectivity. Thus:

$$(a)\qquad (P_nQ_n;\ \cdots)\barwedge(U\cdots)\doublebarwedge A_1(U\cdots)\doublebarwedge(A_n'\cdots)\barwedge(U'\cdots).$$

When U' coincides with E_s, the point A_n' also coincides with E_s, hence U will also coincide with E_s, consequently one point of the couple of elements of the involution $P_nQ_n \cdots$ coincides with E_s. Thus in the projective one-to-two correspondence established by (a) between the points of the lines OA' and s the point E_s is a united element, hence $U'P_n$, $U'Q_n$ envelop a conic tangent to s, the point of contact being the trace on s of the conjugate of OA' in (I), i.e., the line projecting from O the third point of intersection of Γ_n with AA_n'. Thus: *The double lines of the points determined by the cubics of the pencil* Γ *on a line passing through a basic point, envelop a conic tangent to the line considered.*

When the lines OP_n, OQ_n coincide with the lines OB, OC, the point A_n' and hence the point U' will coincide with O, consequently: *The conic is tangent to the lines projecting, from the double point, the other two basic points of* Γ.

17. The points in which a line l through the double point O meets the cubics of the pencil Γ have their tangentials on another line passing through O (6). Hence if l meets one of the cubics of Γ in a point which coincides with its tangential, the same is true about the intersection of l with any other cubic of Γ. Thus: *If a line through the double point meets one of the cubics of the pencil* Γ *in a point of inflexion, it meets all the cubics of the pencil in inflexional points.*

18. Let I_n', I_n'', I_n''' be the three points of inflection of Γ_n. The lines OI_n', OI_n'', OI_n''' meet every cubic of the pencil Γ in three collinear points (14*a*), and all these points are points of inflection (17). Consequently: *The points of inflection of all the cubics of the pencil Γ lie on three straight lines concurring in the double point.*

19. From (18) and (14*b*) it follows: *The inflectional lines of the cubics of the pencil Γ form a pencil having its vertex on the basic line.*

It may be added that this vertex is the harmonic conjugate, with respect to the couple A, A_1, of the trace on ABC of the locus of the tangentials of A (4).

UNIVERSITY OF OKLAHOMA,
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DEFINITION AND ILLUSTRATIONS OF NEW ARITHMETICAL GROUP INVARIANTS.

BY PROFESSOR E. T. BELL.

1. ARITHMETICAL instances of groups are still sufficiently uncommon to make any new occurrence a matter of interest. Many significant group concepts have, of course, been implicit in arithmetic since at least the times of Euler and Gauss, notably in the theories of power residues, the automorphics of binary quadratic forms, and principal genera. More recently Miller has directly applied groups to quadratic residues and other topics.

This note contains the definition and a few illustrations, shorn of algebraic and other details, of certain group invariant relations for arbitrary integers, which are believed to be fundamentally distinct from previous group phenomena in arithmetic. These relations are genuinely arithmetical in that they concern only integers, and they may legitimately be called group relations because they exist only in reference to groups.

The object of this note is merely to call attention to these invariants by exhibiting a few of the simplest; and as several preliminary definitions are required—the subject being new—developments and less obvious examples may be left to another