Kellogg's note gives a simple proof that if this set is closed with respect to continuous functions, it is also closed with respect to summable functions which are not null functions. Hilbert and others have shown that they are closed with respect to continuous functions, and they are therefore closed with respect to the broader class. The interest of the note lies rather in the method of proof than in the results, which are largely already known.

20. In this note Professor Moore discusses the forms of curves that will generate surfaces of constant curvature when rotated by the special rotations leaving a doubly infinite number of planes invariant.

21. In an article in the *Mathematische Abhandlungen* of the Berlin Academy for the year 1857, pages 41–74, Kummer essayed to prove that the relation

$$x^p + y^p + z^p = 0 \tag{1}$$

could not be satisfied in integers, when p is an odd prime not satisfying three given conditions. Based on this result, the conclusion that (1) is impossible for all p's less than 100 was derived by him. In the present paper Mr. Vandiver points out that Kummer made several errors in his argument, which vitiate his results. The paper will appear in the *Proceedings of the National Academy of Sciences*.

F. N. COLE,
Secretary.

STIELTJES DERIVATIVES.

BY PROFESSOR P. J. DANIELL.

THE fundamental theorem for the derivative with respect to a function of limited variation is difficult to prove in the case of several dimensions, and no attempt is made here to consider the most general derived numbers. In place of the method used by the author for one dimension we shall use methods and ideas due to C. de la Vallée Poussin and W. H. Young.*

* C. de la Vallée Poussin, Intégrales de Lebesgue. Paris, 1916, pp. 61–73. W. H. Young, *Proc. London Math. Society* (1914), vol. 13, p. 109. P. J. Daniell, *Transactions Amer. Math. Society* (1918), vol. 19, p. 353.

The theorem to be proved in this paper is:

If in a fundamental interval of several dimensions $\beta(e)$, $F(e)$ are additive functions of sets, $\beta(e)$ non-negative and $F(e)$ absolutely continuous with respect to $\beta(e)$, then there exists a derivative $D_\beta F$ of F with respect to β everywhere except on a set e for which $\beta(e) = 0$; and if E is any set (measurable Borel) contained in the interval,

$$F(E) = \int_E (D_\beta F) d\beta(e).$$

The derivative used is not that obtained from sequences of intervals independent for each point, but a net-derivative.

General considerations. The points p are specified by n coordinates x_r $(r = 1, 2, \cdots, n)$ and an interval consists of a collection of points such that each coordinate lies in a linear continuous interval. In a one-to-one transformation which leaves relative order unchanged, $\beta(e)$, $F(e)$ and $D_\beta F$ will be absolutely invariant, so that the given interval, whatever it may be, can be transformed into one which lies "strictly" within the interval $x_r = 0$ to 1 $(r = 1, 2, \cdots, n)$. By defining $\beta(e)$, $F(e)$ to be 0 for sets not in the transformed interval, we can finally use for the fundamental interval one closed on the left, open on the right, namely

$$0 \leqq x_r < 1 \qquad (r = 1, 2, \cdots, n).$$

$\beta(e)$, $F(e)$ are supposed to be additive in the sense of C. de la Vallée Poussin, that is to say, additive for a countably infinite as well as for a finite number of sets without common points.

Definition 1. If $\beta(e)$, $F(e)$ are additive functions of sets, $F(e)$ is said to be absolutely continuous with respect to $\beta(e)$, if $F(e) = 0$ for all sets e for which $\beta(e) = 0$.

Divide the interval $0 \leqq x_r < 1$ $(r = 1, 2, \cdots, n)$ into 2^{ni} equaLsubintervals (of order i). These will be of the type

$$(m_r - 1)2^{-i} \leqq x_r < m_r 2^{-i} \qquad (r = 1, 2, \cdots, n),$$

where, for each r, m_r is an integer between 1 and 2^i inclusive. Define the function

$$f_i(p) = F(e)/\beta(e),$$

where e is the interval of order i to which p belongs. Throughout the paper we make the convention that when $\beta(e) = 0$, and consequently $F(e) = 0$, the value 0 shall be assigned to the meaningless symbol $F(e)/\beta(e)$. Let $h_i(p)$ be, for each p, the upper bound of $f_{i+t}(p)$, and $h(p)$ the limit of the monotone sequence $h_i(p)$. Then $h(p)$ is the upper limit of the sequence $f_i(p)$.

Definition 2. The upper limit $h(p)$ and the corresponding lower limit $g(p)$ are called the upper and lower net-derivatives of $F(e)$ with respect to $\beta(e)$.

Fundamental lemma. If on a set e measurable Borel $h(p) \geqq s$, then

$$F(e) \geqq s\beta(e).$$

We use the notation $e(f, s)$ to denote the set of points for which $f(p) > s$. $f_i(p)$ is constant over each of the intervals of order i, so that $e_i = e(f_i, s)$ consists of a set of intervals each of which satisfies the inequality above, i.e., $F(e_i) \geqq s\beta(e_i)$. Let $e_{i,\,t} = e_i + e_{i+1} + \cdots + e_{i+t-1}$. Then if the inequality is true for $e_{i,\,t}$ it is true for $e_{i,\,t+1}$. We may write $e_{i,\,t+1} = e_{i,\,t} + e'$, where $e' = e_{i+t} \cdot Ce_{i,\,t}$. $Ce_{i,\,t}$ can be divided into a finite number of intervals of order $i + t$, and all of these which belong to e_{i+t} satisfy the inequality. But $e_{i,\,t}$ and e' have no common point, so that

$$F(e_{i,\,t+1}) - s\beta(e_{i,\,t+1}) = F(e_{i,\,t}) - s\beta(e_{i,\,t}) + F(e') - s\beta(e') \geqq 0.$$

By successive induction the inequality is proved for all $e_{i,\,t}$. If $E_i = \lim e_{i,\,t}$ as $t \doteq \infty$, since $F(e)$, $\beta(e)$ are additive, $F(E_i) = \lim F(e_{i,\,t})$, $\beta(E_i) = \lim \beta(e_{i,\,t})$ and $F(E_i) \geqq s\beta(E_i)$. Now $E_i = e(h_i, s)$; for if at a point p, $h_i > s$ then for some t $f_{i+t} > s$ and p belongs to E_i; while if $h_i(p) \leqq s$, $f_{i+t} \leqq s$ for all t and p belongs to CE_i.

Let $R(s)$ be the set common to all $E_i(s)$ $(i = 1, 2, \cdots)$ and $D(s) = \lim R(s - \epsilon)$ as $\epsilon \doteq 0$, $\epsilon > 0$; then $D(s)$ is the set of points for which $h(p) \geqq s$. For if, at p, $h \geqq s$, then for any $\epsilon > 0$, $h_i > s - \epsilon$ for all i; consequently p belongs to $R(s - \epsilon)$ for any ϵ and therefore to $D(s)$. On the contrary if $h < s$, there is some $\epsilon > 0$ (p fixed) such that $h < s - 2\epsilon$ and a number n can be found so that if $i \geqq n$, $h_i < h + \epsilon < s - \epsilon$. This point p does not belong to $R(s - \epsilon)$ and cannot belong to $D(s)$. But

$$\begin{aligned} F[R(s - \epsilon)] &= \lim F[E_i(s - \epsilon)] \text{ as } i \doteq \infty, \\ &\geqq (s - \epsilon) \lim \beta[E_i(s - \epsilon)] = (s - \epsilon)\beta[R(s - \epsilon)], \\ &\geqq s\beta[R(s)] - \epsilon\beta(I), \end{aligned}$$

where I is the fundamental interval, since $\beta(e)$ is non-negative. Then in the limit as $\epsilon \doteq 0$, $F(D) \geqq s\beta(D)$. When the original interval is replaced by any sub-interval of the type considered the method of reasoning and the function h are unchanged, so that if e is any such interval or finite sum of such intervals without common points

$$F(eD) \geqq s\beta(eD).$$

The inequality will be maintained at successive summations (without common points) and limiting processes. But any set measurable Borel can be obtained from the meshes of the net by these processes, so that if e is measurable Borel, $F(eD) \geqq s\beta(eD)$. This is the required lemma with only a difference in expression.

Conclusion. We are now in a position to prove the main theorem. If B is a set contained in D, e any set $eB = eB \cdot D$. In particular the set B where $h(p) < t$, i.e., where the lower limit of $-f_i(p) > -t$, is included in the set where the upper limit of $-f_i(p) \geqq -t$. On putting $-F$ in place of F and therefore $-f_i$ in place of f_i, the lemma shows that $-F(eB) \geqq -t\beta(eB)$, or $F(eB) \leqq t\beta(eB)$. If $A = BD$, i.e., the set where $s \leqq h(p) < t$, combining the results,

$$s\beta(eA) \leqq F(eA) \leqq t\beta(eA).$$

If on a set e, the upper limit of $f_i(p) = h(p) = +\infty$, then $h(p) > s$ for all s and $s\beta(e) \leqq F(e)$ so that $\beta(e) = 0$, and consequently $F(e) = 0$. The same can be proved of the set where $h = -\infty$. Thus $h(p)$ is finite everywhere except on a set of β-measure 0. Also from the above inequality and the definition of the generalized Stieltjes integral, $h(p)$ is summable and if E is any set measurable Borel

$$F(E) = \int_E h(p) d\beta(e).$$

By exactly the same reasoning, if $g(p)$ is the lower limit of $f_i(p)$, $g(p)$ is finite "nearly everywhere (β)," is summable and

$$F(E) = \int_E g(p) d\beta(e).$$

The difference of the two integrals is 0, while $h \geqq g$ so that $h = g$, i.e., the sequence f_i converges to a single limit $f(p)$

$= h(p) = g(p)$, nearly everywhere (β). This limiting function may be called the net-derivative $D_\beta F$. The main theorem is now proved, since $D_\beta F$ exists and is finite nearly everywhere (β) and if E is measurable Borel,

$$F(E) = \int_E D_\beta F d\beta(e).$$

The proof does not depend on the special method of division; it is sufficient if the "meshes" are divided progressively and if any interval is "Borel measurable" using the meshes as a basic family. The net-derivatives obtained from a finite or countably infinite number of different nets will be the same except on a set of β-measure 0. For if two such net-derivatives are f, k then the sets where $f \geqq k$, and where $f < k$ are each measurable Borel and the integral of $f - k$ with respect to β is 0 over each. But the sum of a countably infinite number of sets of zero measure is itself of zero measure. A proof for *all* net-derivatives considered together is lacking but would be desirable. The method of this paper can easily be extended to derivatives in a countably infinite number of dimensions, corresponding to integrals of the same type exhibited by the author.* For this purpose it is convenient, if not necessary, to commence the sub-division in one dimension after another so that a typical mesh of order i would be

$$(m_r - 1)2^{-(i+1-r)} \leqq x_r < m_r 2^{-(i+1-r)} \quad (r = 1, 2, \cdots, i),$$
$$0 \leqq x_r < 1 \quad (r = i+1, i+2, \cdots),$$

where m_r is an integer between 1 and 2^{i+1-r} $(r = 1, 2, \cdots, i)$ inclusive.

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* P. J. Daniell, *Annals of Mathematics* (1919), vol. 21, p. 30.