3. Solution of equations satisfying the Lipschitz condition by the method of successive approximations.
4. Properties of the solutions.
5. Extension of the solution to a boundary of the region for which the equations are defined.
6. Solution by the Cauchy-Lipschitz method.
7. Solutions of infinite systems of linear differential equations having constant coefficients.
8. Solutions of infinite systems of linear differential equations having periodic coefficients.

LECTURE V. APPLICATIONS OF FUNCTIONS OF INFINITELY MANY VARIABLES.

1. Hill's problem of the motion of the lunar perigee.
2. Solutions of linear differential equations in the vicinity of singular points.
3. The determination of the moon's variational orbit.
4. Determination of periodic solutions of certain finite systems of differential equations.
5. The dynamics of a certain type of infinite universe.

At the close of the colloquium, Professor E. B. Van Vleck expressed the appreciation of those present for the excellence of the lectures, and tendered the thanks of the American Mathematical Society to the University of Chicago for the generous provision it had made for the colloquium, and for the welfare of the participants. An appropriate reply was made by Professor E. H. Moore.

W. A. HURWITZ.

NOTE ON VELOCITY SYSTEMS IN CURVED SPACE OF N DIMENSIONS.

BY PROFESSOR JOSEPH LIPKA.

(Read before the American Mathematical Society April 24, 1920.)

§ 1. *Introduction.*

IN a previous paper,* the author gave a complete geometric characterization of the families of curves (termed natural

* "Natural families of curves in a general curved space of n dimensions," *Trans. Amer. Math. Society*, vol. 13 (1912), pp. 77–95. We shall hereafter refer to this paper as "Natural families."

families) defined as the extremals connected with variation problems of the form

$$(1) \qquad \int F ds = \text{minimum},$$

where F is any point function and ds is the element of arc in the space considered. Such a system consists of $\infty^{2(n-1)}$ curves, one through each point in each direction. Among the dynamical systems whose determination leads to an integral of this form we may mention: (1) the trajectories in a conservative field of force for a given constant of energy h, where $F = \sqrt{W + h}$, W being the work function (negative potential); (2) the brachistochrones under conservative forces, $F = 1/\sqrt{W + h}$; (3) the forms of equilibrium of a homogeneous, flexible, inextensible string acted on by conservative forces (general catenaries), $F = W + h$; the paths of light in an isotropic medium, $F = \nu$, the variable index of refraction.

The complete characteristic geometrical properties of a natural family in any curved space, V_n, are:*

(A_1) The locus of the centers of geodesic curvature of the ∞^{n-1} curves which pass through any point of V_n is a euclidean space of $n - 1$ dimensions (S_{n-1}).

(A_2) The osculating geodesic surfaces (V_2's) at any point of V_n form a bundle of surfaces, i.e., all contain a fixed direction (and hence the geodesic in that direction) which is normal to the S_{n-1} of property A_1.

(B) The n directions at any point of V_n, in which, as a consequence of property A (i.e., A_1 and A_2), the osculating geodesic circles (circles of constant geodesic curvature) hyperosculate the curves of the given family, are mutually orthogonal.

Now, property A alone completely characterizes a much wider class of curves, designated as a velocity system.† We have pointed out several dynamical problems which lead to a system of curves characterized by properties A and B. It is the purpose of this note to point out a dynamical problem which leads to the more general system of curves characterized by property A alone.‡

* "Natural families," p. 78.

† This was designated as a system of type (G) in "Natural families," pp. 85–86.

‡ See the discussion of the problem for a euclidean space of 3 dimensions by E. Kasner, Princeton Colloquium Lectures, p. 42.

§ 2. *Differential Equations of Velocity Systems.*

Let us express our problem analytically. If the element of arc length in a general curved space V_n is given by*

$$ds^2 = \sum_{ik} a_{ik} dx_i dx_k \tag{2}$$

and we use s as the parameter along our curves, the differential equations of any natural system (characterized by properties A and B) are†

$$x_i'' + \sum_{\lambda\mu} \begin{Bmatrix} \lambda\mu \\ i \end{Bmatrix} x_\lambda' x_\mu' = \sum_l \frac{\partial(\log F)}{\partial x_l}(A_{il} - x_i' x_l') \qquad (i = 1, 2, \cdots, n), \tag{3}$$

where A_{il} denotes the minor of a_{il} in the determinant $a = |a_{\lambda\mu}|$ divided by a itself, and we have used the Christoffel symbols

$$\begin{Bmatrix} \lambda\mu \\ i \end{Bmatrix} = \sum_l A_{il} \begin{bmatrix} \lambda\mu \\ l \end{bmatrix}; \qquad \begin{bmatrix} \lambda\mu \\ l \end{bmatrix} = \sum_i a_{il} \begin{Bmatrix} \lambda\mu \\ i \end{Bmatrix}; \tag{4}$$

$$\begin{bmatrix} \lambda\mu \\ l \end{bmatrix} = \frac{1}{2}\left(\frac{\partial a_{\lambda l}}{\partial x_\mu} + \frac{\partial a_{\mu l}}{\partial x_\lambda} - \frac{\partial a_{\lambda\mu}}{\partial x_l} \right).$$

On the other hand, the differential equations of a velocity system (characterized by property A) are‡

$$x_i'' + \sum_{\lambda\mu} \begin{Bmatrix} \lambda\mu \\ i \end{Bmatrix} x_\lambda' x_\mu' = \sum_l \phi_l (A_{il} - x_i' x_l') \qquad (i = 1, 2, \cdots, n), \tag{5}$$

where the ϕ's are arbitrary point functions. This system will reduce to a natural system if

$$\phi_l = \frac{\partial(\log F)}{\partial x_l} \qquad (l = 1, 2, \cdots, n), \tag{6}$$

i.e., the ϕ's are the partial derivatives with respect to x of a single function. This is the analytic equivalent of property B.

---

* We write only Σ_{ik} and understand that the summation is to be carried out from 1 to n for each of the indicated subscripts.

† "Natural families," p. 80. Throughout this paper, primes refer to total derivatives with respect to arc length s, while dots refer to total derivatives with respect to time t.

‡ "Natural families," p. 85.

§ 3. *Dynamical Interpretation of Velocity System.*

Let us consider the motion of a particle in a curved n-space V_n under any positional forces. We start with the Lagrangian equations of motion,*

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_l}\right) - \frac{\partial T}{\partial x_l} = X_l \quad (l = 1, 2, \cdots, n), \tag{7}$$

where T is the kinetic energy, given by

$$T = \tfrac{1}{2}\sum_{ik} a_{ik}\dot{x}_i\dot{x}_k, \tag{8}$$

and the X's are the components of force given as functions of the coordinates $x_1, x_2, \cdots, x_n$. Equations (7) may be expanded as

$$\begin{aligned} X_l &= \frac{d}{dt}\left(\sum_k a_{lk}\dot{x}_k\right) - \frac{1}{2}\sum_{ik}\frac{\partial a_{ik}}{\partial x_l}\dot{x}_i\dot{x}_k, \\ &= \sum_k a_{lk}\ddot{x}_k + \sum_{ik}\frac{\partial a_{lk}}{\partial x_i}\dot{x}_i\dot{x}_k - \frac{1}{2}\sum_{ik}\frac{\partial a_{ik}}{\partial x_l}\dot{x}_i\dot{x}_l, \\ &= \sum_k a_{lk}\ddot{x}_k + \sum_{ik}\begin{bmatrix} ik \\ l \end{bmatrix}\dot{x}_i\dot{x}_k. \end{aligned}$$

Multiplying by A_{ml}, summing with respect to l, and employing

$$\sum_i a_{ik}A_{il} = \begin{Bmatrix} 0, \text{ if } k \neq l \\ 1, \text{ if } k = l \end{Bmatrix}, \tag{9}$$

we get

$$\ddot{x}_m + \sum_{ik}\begin{Bmatrix} ik \\ m \end{Bmatrix}\dot{x}_i\dot{x}_k = \sum_l A_{ml}X_l \quad (m = 1, 2, \cdots, n). \tag{10}$$

These equations give us the components of acceleration along a curve as functions of the coordinates and the components of velocity.

Since the velocity along the curve is given by

$$\dot{s}^2 = \sum_{ik} a_{ik}\dot{x}_i\dot{x}_k, \tag{11}$$

we may, by differentiation, get the acceleration along the path; thus

* See E. T. Whittaker, Analytical Dynamics, p. 39.

$$
\begin{aligned}
\dot s \ddot s &= \sum_{ik} a_{ik}\dot x_i \ddot x_k + \frac{1}{2}\sum_{ikr}\frac{\partial a_{ik}}{\partial x_r}\dot x_i \dot x_k \dot x_r \\
&= \sum_{ik} a_{ik}\dot x_i\Big(\sum_l A_{kl}X_l - \sum_{\alpha\beta}\begin{Bmatrix}\alpha\beta\\ k\end{Bmatrix}\dot x_\alpha \dot x_\beta\Big) + \frac{1}{2}\sum_{ikr}\frac{\partial a_{ik}}{\partial x_r}\dot x_i \dot x_k \dot x_r \\
&= \sum_l \dot x_l X_l - \sum_{i\alpha\beta}\begin{bmatrix}\alpha\beta\\ i\end{bmatrix}\dot x_i \dot x_\alpha \dot x_\beta + \frac{1}{2}\sum_{i\alpha\beta}\frac{\partial a_{i\alpha}}{\partial x_\beta}\dot x_i \dot x_\alpha \dot x_\beta \\
&= \sum_l \dot x_l X_l,
\end{aligned} \tag{12}
$$

the reductions being accomplished by using (4) and (9).

As a first step in getting the differential equations of the trajectories, we have

$$x_m' = \frac{\dot x_m}{\dot s}; \qquad x_m'' = \frac{\dot s \ddot x_m - \dot x_m \ddot s}{\dot s^3};$$

and using (10) and (12), we get

$$x_m'' = \frac{1}{\dot s^2}\sum_l A_{ml}X_l - \sum_{ik}\begin{Bmatrix}ik\\ m\end{Bmatrix}x_i'x_k' - \frac{1}{\dot s^2}x_m'\sum_l x_l'X_l,$$

or

$$x_m'' + \sum_{ik}\begin{Bmatrix}ik\\ m\end{Bmatrix}x_i'x_k' = \frac{1}{\dot s^2}\sum_l X_l(A_{ml} - x_m'x_l') \qquad (m = 1, 2, \cdots, n). \tag{13}$$

To get the differential equations of the trajectories we should have to eliminate the speed $\dot s$; this would lead to a set of equations of the third order representing ∞^{2n-1} curves. But, for our purpose, we need not go any further. Equations (13) hold for any trajectory, and along this the speed $\dot s$ varies from point to point. Now if in (13) we replace $1/\dot s^2$ by a constant c, we get

$$x_m'' + \sum_{ik}\begin{Bmatrix}ik\\ m\end{Bmatrix}x_i'x_k' = c\cdot\sum_l X_l(A_{ml} - x_m'x_l') \qquad (m = 1, 2, \cdots, n), \tag{14}$$

a set of differential equations of the second order representing a system of $\infty^{2(n-1)}$ curves (called a velocity system) one through each point in each direction. This system may therefore be defined dynamically as follows:

A curve is a velocity curve corresponding to the speed $\dot{s}_0$ *if a particle starting from a point of such a curve and in the direction of the curve and with that speed describes a trajectory osculating the curve.*

Now, we note that equations (14) are, with a change in subscripts, exactly equations (5), where

$$(15) \qquad \phi_l = cX_l = \frac{1}{\dot{s}^2}X_l \qquad (l = 1, 2, \cdots, n).$$

We have thus formulated a dynamical problem which leads to the system of curves (called a velocity system) characterized geometrically by property A. For each constant value assigned to the speed $\dot{s}$, we get a velocity system, and the totality of ∞^1 systems obtained by varying $\dot{s}$ constitute a complete velocity system of ∞^{2n-1} curves in V_n.

§ 4. *Velocity Systems and Natural Systems.*

Velocity systems are not in general systems of trajectories, brachistochrones, or catenaries, but if the field of force is conservative, then

$$(16) \qquad X_l = \frac{\partial W_1}{\partial x_l} \qquad (l = 1, 2, \cdots, n),$$

where W_1 is the work function defining the field, and, as pointed out in § 2, the velocity system corresponding to a speed $\dot{s}_0$ becomes a natural system defined by equation (1) or by the point function F, where by (6)

$$\phi_l = \frac{\partial(\log F)}{\partial x_l},$$

and by (15)

$$c\frac{\partial W_1}{\partial x_l} = \frac{\partial(\log F)}{\partial x_l};$$

hence

$$(17) \qquad F = e^{W_1/\dot{s}^2}$$

On the other hand, in a conservative field of force with work function W_2 and given constant of energy h, the natural system defined by equation (1) or by the point function F is a system of

$$
(18)\qquad \begin{array}{ll} \text{trajectories,} & \text{if } F = \sqrt{W_2 + h}, \\ \text{brachistochrones,} & \text{if } F = 1/\sqrt{W_2 + h}, \\ \text{catenaries,} & \text{if } F = W_2 + h. \end{array}
$$

By comparison of (17) and (18), we may now state:

A velocity system for the speed $\dot{s}_0$ in a conservative field with work function W_1 is a system of (1) *trajectories,* (2) *brachistochrones,* (3) *catenaries for the constant of energy h in a conservative field with work function W_2, where*

$$
(1)\ W_2 = e^{2W_1/\dot{s}_0^2} - h, \qquad (2)\ W_2 = e^{-(2W_1/\dot{s}_0^2)} - h,
$$

$$
(3)\ W_2 = e^{W_1/\dot{s}_0^2} - h.
$$

Since $W_1 =$ constant gives $W_2 =$ constant, the two fields have the same equipotential hypersurfaces and the same lines of force.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
February, 1920.

AUGUSTUS DE MORGAN ON DIVERGENT SERIES.

BY PROFESSOR FLORIAN CAJORI.

(Read before the San Francisco Section of the American Mathematical Society April 10, 1920.)

SEVERAL English mathematicians writing in the second quarter of the nineteenth century disapproved of the banishment of divergent series which had been brought about by the followers of A. L. Cauchy and N. H. Abel. These protests were unheeded, doubtless because they were not accompanied by indications disclosing how divergent series could be used with safety. There was one exception, however: Augustus De Morgan reached results which, had they been followed up promptly, might have re-introduced divergent series thirty years earlier than was actually the case. De Morgan's researches have been overlooked in historical statements, except by H. Burkhardt,* who, however, missed the parts of De Morgan which foreshadow a new theory.

* H. Burkhardt "Ueber den Gebrauch divergenter Reihen in der Zeit von 1750–1860," *Math. Annalen*, vol. 70 (1911), pp. 169–206. This article contains much minute information regarding many writers.