THE OCTOBER MEETING OF THE SAN FRANCISCO SECTION.

THE thirty-sixth regular meeting of the San Francisco Section was held at the University of California on Saturday, October 23. The chairman of the Section, Professor Blichfeldt, presided at the early part of the meeting; Professor Lehmer presided at the latter part. The attendance was twenty-two, including the following fifteen members of the Society:

Professor R. E. Allardice, Professor B. A. Bernstein, Professor H. F. Blichfeldt, Professor Florian Cajori, Professor M. W. Haskell, Professor Frank Irwin, Professor D. N. Lehmer, Professor W. A. Manning, Professor H. C. Moreno, Dr. F. R. Morris, Professor C. A. Noble, Professor T. M. Putnam, Dr. Pauline Sperry, Dr. S. E. Urner, Dr. A. R. Williams.

The following officers were elected for the year: chairman, Professor D. N. Lehmer; secretary, Professor B. A. Bernstein; programme committee, Professors H. F. Blichfeldt, W. A. Manning, B. A. Bernstein.

The dates of the next two meetings were provisionally fixed as April 9, 1921, and October 22, 1921.

The following papers were presented:

(1) Professor D. N. LEHMER: "On inverse ternary continued fractions."

(2) Professor M. W. HASKELL: "Curves autopolar with respect to two conics."

(3) Professor FLORIAN CAJORI: "Historical note on notations for ratio and proportion."

(4) Professor E. T. BELL: "The elliptic modular equation of the third order and the form $x^2 + 3y^2$."

(5) Professor E. T. BELL: "The reversion of class number relations and the total representation of an integer as a sum of square or triangular numbers."

(6) Professor E. T. BELL: "Class numbers and the form $xy + yz + zx$."

(7) Professor E. T. BELL: "Singly infinite class number relations."

(8) Professor E. T. BELL: "On recurrences for sums of divisors."

(9) Mr. H. W. BRINKMANN: "The group characteristics of the ternary linear fractional group and of various other groups."

(10) Professor H. F. BLICHFELDT: "Notes on geometry of numbers."

Mr. Brinkmann was introduced by Professor Manning. In the absence of the author the papers of Professor Bell were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. A ternary continued fraction is defined by means of its partial quotient pairs $(p_1, q_1; p_2, q_2; p_3, q_3; \cdots; p_n, q_n)$, the successive convergent sets (A_n, B_n, C_n) being solutions of the difference equation $u_n = q_n u_{n-1} + p_n u_{n-2} + u_{n-3}$, where the initial values for A_n, B_n, C_n are respectively, 1, 0, 0; 0, 1, 0; 0, 0, 1. (See the *Proceedings of the National Academy of Sciences*, volume 4, pages 360–364, December, 1918.)

It is of importance in the theory of cubic irrationalities to know what relations must exist among the partial quotient pairs in order that the two inverse ternary continued fractions $(p_1, q_1; p_2, q_2; \cdots; p_n, q_n)$, $(p_n, q_n; p_{n-1}, q_{n-1}; \cdots; p_1, q_1)$ should have the same characteristic cubic, the characteristic cubic being defined by the equation

$$\begin{vmatrix} A_{n-2} - \rho & B_{n-2} & C_{n-2} \\ A_{n-1} & B_{n-1} - \rho & C_{n-1} \\ A_n & B_n & C_n - \rho \end{vmatrix} = 0.$$

Professor Lehmer finds, besides the obvious case where the pairs read the same backward and forward, that the cubic is the same for both fractions if $p_i = \alpha t + \beta$, $q_i = \gamma t + \delta$, where $\alpha, \beta, \gamma, \delta$ are any fixed positive or negative integers or zero, while t is a variable parameter.

2. Professor Haskell described the configuration of four mutually autopolar conics, and gave a method for finding an unlimited number of higher plane curves autopolar with respect to two conics, when the latter are autopolar with respect to each other.

3. Professor Cajori notes that the colon was first used for ratio by the astronomer Vincent Wing in 1651. The forerunner

of Oughtred's notation $A \cdot B :: C \cdot D$ to signify $A : B = C : D$ is found in Billingsley's Euclid, 1570, where we find $9 \cdot 6 : 12 \cdot 8$. With Billingsley, the dot and colon were symbols of punctuation which were not yet limited to specific arithmetical use. Isaac Barrow wrote $A \cdot B + C \cdot D$ to signify the "compounding of ratios," i.e., $A : B$ times $C : D$. John Wallis opposed this practice. Sometimes, in the same equation, Barrow used $+$ to indicate the multiplication of ratios as well as the addition of terms.

4. In Professor Bell's first paper it is shown that the modular equation for the transformation of the third order in elliptic functions is implied by a theorem relating to even functions of two variables, the arguments of the functions being linear functions of the integers which represent an arbitrary integer in the form $x^2 + 3y^2$. The theorem admits of easy extension (for the appropriate quadratic forms), to the modular equation of the nth order, and takes particularly simple forms when n is a prime $4k + 1$, $12k + 7$ or the triple of a prime $12k + 1$. The paper will be published in the *Bulletin* of the Greek Mathematical Society.

5. In Professor Bell's second paper it is shown that the class number relations of Kronecker, Hermite and others may be reversed so as to give the class number for a negative determinant explicitly in terms of the total number of representations of certain integers each as a sum of square or triangular numbers, and further it is shown that each of a pair of such expressions (a relation and its inversion) implies the other. Recurrences for the computation of the new functions relating to total numbers of representations are given. These bear a striking resemblance to those for the class numbers.

6. In Professor Bell's third paper three general class number relations, each of which admits of specialization in an infinity of ways, are derived and their connection with the arithmetical form $xy + yz + zx$ considered. Each relation involves class numbers and an arbitrary even function of a single variable. On particularizing the function in various ways several special class number theorems are found at once, the simplest of which is: the total number of representations of n in the form

$xy + yz + zx$ for which $x, y, z > 0$, is $3[G(n) - 1]$ when and only when n is prime, $G(n)$ being the whole number of classes of binary quadratic forms for the determinant $-n$. The next simplest cases are famous class number relations due either to Kronecker or Hermite; the next are of the Liouville types, and thence onwards the special consequences are class number relations of kinds not hitherto stated. The paper will appear shortly in the *Tôhoku Mathematical Journal.*

7. A set of seventeen closely interrelated class number formulas of the kind described in Professor Bell's third paper is derived. In several ways the set is complete, no more results of the same general sort being implicit in the analysis. On specializing the arbitrary even functions involved in the most obvious ways, all of Kronecker's and Hermite's formulas drop out as the simplest cases, also some of those due to Liouville and Humbert. The method used was explained in the first part of a paper presented to the Society in December, 1918, which will appear in the *Transactions*.

8. A considerable section of Chapter X (volume I) of Dickson's History of the Theory of Numbers is devoted to divisor recurrences obtained from the expansions of elliptic functions. Professor Bell shows in this paper that any such recurrence is a very special case of a general relation between the divisors concerned, and specific application is made (in one of the illustrations) to well-known formulas of Halphen and Glaisher. A curious lacuna is observed: none of the formulas is sufficient for the calculation by recurrence alone of the $2r$th ($r = 0, 1, 2, \cdots$) powers of *all* the divisors, although the like may easily be done for the $(2r - 1)$th powers.

9. Mr. Brinkmann determines the group characteristics of the group of all ternary linear fractional substitutions of determinant unity whose coefficients are marks of any Galois field, and also the group characteristics of the group of all ternary linear fractional substitutions of non-vanishing determinant.

Further, the group characteristics of all primitive permutation groups of degree less than or equal to fifteen are determined, so far as they are not known already.

10. In Minkowski's development of geometry of numbers

(cf. this BULLETIN, volume 25 (1919), page 449) the following theorem is fundamental: if we designate by a Minkowski surface in R_n a finite surface in space of n dimensions, having as its chief characteristic a center of symmetry toward which it is nowhere convex (cf. l. c. for specific definition), then a Minkowski surface in R_n and of volume $\geqq 2^n$ will contain at least three distinct lattice points (i. e., points whose coordinates are integers) if its center is a lattice point. In order to extend the usefulness of the geometry of numbers, Professor Blichfeldt has amplified this theorem to read as follows: (1) a Minkowski surface in R_n of volume $\geqq 2^n k$ and whose center is a lattice point, must contain more than $k - 1$ distinct pairs of lattice points in addition; (2) a Minkowski surface in R_n which contains k lattice points, its center being one, must have a volume $> (k - n)/n!$, if these k points do not all lie on a linear R_{n-1}. Some applications of this theorem were presented.

B. A. BERNSTEIN,
Secretary of the Section.

AN IMAGE IN FOUR-DIMENSIONAL LATTICE SPACE OF THE THEORY OF THE ELLIPTIC THETA FUNCTIONS.

BY PROFESSOR E. T. BELL.

(Read before the San Francisco Section of the American Mathematical Society June 18, 1920.)

1. In his memoir on "Rotations in space of four dimensions"* Professor Cole defined a system of four mutually orthogonal lineoids *yzw, xzw, xyw, xyz* (which we shall denote by X, Y, Z, W respectively) through a point O, the four lines and six planes determined by these, and with reference to this system found the transformations into itself of a sphere S with center at O. Henceforth we assume the radius of S to be $\sqrt{n}$, where n is an integer > 0. From this system we shall derive an image of the theory of the elliptic theta func-

* *Amer. Jour. of Math.*, vol. 12 (1890), p. 191.