(cf. this BULLETIN, volume 25 (1919), page 449) the following theorem is fundamental: if we designate by a Minkowski surface in R_n a finite surface in space of n dimensions, having as its chief characteristic a center of symmetry toward which it is nowhere convex (cf. l. c. for specific definition), then a Minkowski surface in R_n and of volume $\geqq 2^n$ will contain at least three distinct lattice points (i. e., points whose coordinates are integers) if its center is a lattice point. In order to extend the usefulness of the geometry of numbers, Professor Blichfeldt has amplified this theorem to read as follows: (1) a Minkowski surface in R_n of volume $\geqq 2^n k$ and whose center is a lattice point, must contain more than $k - 1$ distinct pairs of lattice points in addition; (2) a Minkowski surface in R_n which contains k lattice points, its center being one, must have a volume $> (k - n)/n!$, if these k points do not all lie on a linear R_{n-1}. Some applications of this theorem were presented.

B. A. BERNSTEIN,
Secretary of the Section.

AN IMAGE IN FOUR-DIMENSIONAL LATTICE SPACE OF THE THEORY OF THE ELLIPTIC THETA FUNCTIONS.

BY PROFESSOR E. T. BELL.

(Read before the San Francisco Section of the American Mathematical Society June 18, 1920.)

1. In his memoir on "Rotations in space of four dimensions"* Professor Cole defined a system of four mutually orthogonal lineoids yzw, xzw, xyw, xyz (which we shall denote by X, Y, Z, W respectively) through a point O, the four lines and six planes determined by these, and with reference to this system found the transformations into itself of a sphere S with center at O. Henceforth we assume the radius of S to be $\sqrt{n}$, where n is an integer > 0. From this system we shall derive an image of the theory of the elliptic theta func-

* *Amer. Jour. of Math.*, vol. 12 (1890), p. 191.

tions by considering the reflexions of certain point configurations $C, C', C'', C''', C^{\text{iv}}$ lying upon S with respect to O and the bisectors of the angles between X, Y, Z, W, a bisector of an angle between two lineoids being defined as a locus of points equidistant from the two. The space about O is latticed by four systems of lineoids parallel respectively to X, Y, Z, W, the successive lineoids in each system being at unit distances apart. We shall call this the unit lattice L. Any point all of whose coordinates are integers belongs to L; and conversely L contains only such points. Any integer > 0 being in several ways a sum of four integral squares, S always contains points of L, and these are symmetrical in pairs with respect to O.

Denote by L' the lattice containing all those points and only those whose coordinates are $(4a, 4b, 4c, 4d)$, where a, b, c, d take all integral values (including zero) from $-\infty$ to $+\infty$, and by $\alpha\beta\gamma\delta$ the lattice derived from L' by successive translations of L' through distances $\alpha, \beta, \gamma, \delta$ parallel respectively to X, Y, Z, W, where $\alpha, \beta, \gamma, \delta$ are integers $\geqq 0$, so that L' is 0000. There clearly are in all precisely 256 distinct lattices $\alpha\beta\gamma\delta$, each of which is contained in L, and these may be represented by symbols $\alpha'\beta'\gamma'\delta'$, where $\alpha', \beta', \gamma', \delta'$ are the positive residues mod 4 of $\alpha, \beta, \gamma, \delta$. For brevity we assign current numbers to a special set of 64 contained in the 256. Only the $\alpha'\beta'\gamma'\delta'$ wherein α', β' are both even or both odd are required in the sequel, and likewise for γ', δ'. The requisite half-symbols $\alpha'\beta', \gamma'\delta'$ are therefore 00, 02, 11, 13, 20, 22, 31, 33. Write

$$20, 22, 02, 00 \equiv 2, 4, 6, 8,$$

$$11, 13, 31, 33 \equiv 1, 3, 5, 7,$$

respectively. In this notation the lattice 0222 is 64; the current number of 3320 is 72; that of 1311 is 31; 3100 is 58; 0231 is denoted by 65, etc. To signify that all the points of a certain configuration C belong to one of these lattices, say to ij, we give C the corresponding double suffix, C_{ij}. The theory of the theta functions is formally equivalent to the symmetries of certain C_{ij} lying upon S. By the formal equivalence of A, B we mean that each implies the other.

2. Consider first any point P lying within (not on any of X, Y, Z, W) any one T of the sixteen right tetraedral angles into which space is partitioned by X, Y, Z, W. Bisect the

angles which any one of X, Y, Z, W, say X, makes with the remaining three, and denote by Y', Z', W' those parts of the bisectors which lie within T, and by $U', V', U'', V'', U''', V'''$ the like parts of the bisectors of the angles between the three pairs of opposite pairs X, Y and Z, W; X, Z and Y, W; X, W and Y, Z, viz., U' is the bisector for X, Y; V' for Y, Z, etc. (We have chosen the internal bisectors with respect to the angles of T.) Reflect P in Y', reflect the image in Z', and reflect this image in W', getting finally the point P_1. In whatever order the three successive reflexions are performed, it is clear that the same P_1 is reached. From P_1 in the same way derive P_2, from P_2 similarly P_3, and from P_3 in the same way P_4. Then $P_4 \equiv P$. Second, if P lies on at least one of X, Y, Z, W, we avoid ambiguities (of sign) by requiring the reflexions to be performed so that the signs of the coordinates of P are unchanged; e.g., the signs being $(++--)$ are to be the same before and after reflexion. Reflect P in O, getting P_0; join the centroid Π of P_0, P_1, P_2, P_3 to O, and produce $O\Pi$ through Π to cut S in P', which point we shall call the mate of P. Reflect P' in U', reflect the image in V', getting P'', called the first skew mate of P. Similarly from P' and U'', V'' get P''', and from U''' and V''', get P^{iv}, the second and third skew mates of P. Note that $P', \cdots, P^{\text{iv}}$ are significant only with respect to the particular T in which P lies. Taking the mates of all points in any configuration C we get its mate C', and similarly for the first, second and third skew mates C'', C''', C^{iv}.

We shall be concerned with two kinds of symmetry about O of C_{ij} and their mates. If A, B are any configurations such that each may be brought into coincidence with the other by reflexions in O of some (or all) of its points, A, B are called images of each other. All those points of any configuration C which are such that no one of them is the reflexion in O of another, form a configuration called the residue of C; and any configurations A_1, B_1 are said to be skew images of each other when their residues coincide.

Finally we need also the idea of lattice configurations with multiple points. Let each point of $C_1, C_2, \cdots, C_r$ belong to the unit lattice. By $C_1 + C_2 + \cdots + C_r, \equiv C$, we mean the configuration which consists of all the points of $C_1, C_2, \cdots, C_r$. A point occurring in precisely s of the C_i is multiple of order s in C; and two coincident configurations are identical when and only when points occupying the same position in both are of equal multiplicities. In particular if A, B are images or skew images of each other, the points in any pair which are regarded as reflexions in O of one another must be of the same multiplicity. To indicate that each point in C is of multiplicity s, we write sC.

We can now give the image of the single theta functions. The geometrical theorems will first be stated, their theta equivalents then pointed out, and the means for passing from one to the other briefly indicated. The geometry can be derived simply from first principles. It is shorter, however, to proceed as in § 4. The eleven images can be compressed into one relating to S and the unit lattice, but the statement is complicated. The eleven exhibit a manifold symmetry recalling that of crystals, which becomes evident when the C_{ij} and their mates and skew mates C_{ij}', C_{ij}'', etc., are written with the suffixes in full, thus, C_{0011}, C_{3320}'. In all that follows the C_{ij}, and therefore also their mates and skew mates, are configurations of points lying on the S of radius $\sqrt{n}$ defined in § 1.

3. The first two theorems concern the case $n \equiv 0 \bmod 4$, and the configurations

$$C_0 \equiv C_{22} + C_{26} + C_{62} + C_{66}, \quad C_2 \equiv C_{24} + C_{28} + C_{64} + C_{68},$$

$$C_4 \equiv C_{44} + C_{48} + C_{84} + C_{88}, \quad C_6 \equiv C_{42} + C_{46} + C_{82} + C_{86};$$

$$C_1 \equiv C_{11} + C_{17} + C_{71} + C_{77}, \quad C_3 \equiv C_{13} + C_{15} + C_{73} + C_{75},$$

$$C_5 \equiv C_{33} + C_{35} + C_{55} + C_{53}, \quad C_7 \equiv C_{31} + C_{37} + C_{51} + C_{57}.$$

THEOREM I. *Each of $C_0 + C_4$, $C_1 + C_5$ is the image of its mate, and each of $C_2 + C_6$, $C_3 + C_7$ is the image of the mate of the other.*

THEOREM II. *The configurations in each of the following pairs are images of each other*:

$$C_0 + C_2 + C_4' + C_6' + C_1 + C_3 + C_5' + C_7',$$

$$C_0' + C_2' + C_4 + C_6 + C_1' + C_3' + C_5 + C_7;$$

$$C_0 + C_2 + C_2' + C_4' + C_1' + C_5 + C_7 + C_7',$$

$$C_0' + C_4 + C_6 + C_6' + C_1 + C_3 + C_3' + C_5'.$$

The next two are for $n \equiv 2 \bmod 4$, and the configurations

$$B_1 \equiv C_{12} + C_{16} + C_{72} + C_{76}, \quad B_3 \equiv C_{34} + C_{38} + C_{54} + C_{58},$$

$$B_5 \equiv C_{14} + C_{18} + C_{74} + C_{78}, \quad B_7 \equiv C_{32} + C_{36} + C_{52} + C_{56}.$$

THEOREM III. *$B_1 + B_3$ is the image of its second skew mate, and $B_5 + B_7$ is the image of its mate.*

THEOREM IV. *$B_1 + B_3'''$ is the image of $B_1''' + B_3$, and $B_5' + B_7$ is the image of $B_5 + B_7'$.*

The next is also for $n \equiv 2 \bmod 4$, and the configurations

$$D_1 \equiv C_{16} + C_{18} + C_{36} + C_{38} + C_{52} + C_{54} + C_{72} + C_{74},$$

$$D_2 \equiv C_{12} + C_{14} + C_{32} + C_{34} + C_{56} + C_{58} + C_{76} + C_{78}.$$

THEOREM V. *The following are skew images of each other:*

$$2D_1 + D_1' + D_2'' + D_2''' + D_2^{\text{iv}},$$

$$2D_2 + D_2' + D_1'' + D_1''' + D_1^{\text{iv}}.$$

4. The foregoing theorems imply the theory of the theta functions. For, if f, g are single-valued functions of four variables existing when each variable takes integral values $\gtreqless 0$, such that

$$f(x, y, z, w) = \quad f(-x, -y, -z, -w),$$

$$g(x, y, z, w) = -g(-x, -y, -z, -w),$$

and otherwise are wholly arbitrary, we may express that A, B are images, that A, B are skew images, by

$$\Sigma f(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \Sigma f(\beta_1, \beta_2, \beta_3, \beta_4),$$

$$\Sigma g(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \Sigma g(\beta_1, \beta_2, \beta_3, \beta_4)$$

respectively, the Σ's extending to all points $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ of A, and $(\beta_1, \beta_2, \beta_3, \beta_4)$ of B. On remarking that if $P \equiv (x_1, x_2, x_3, x_4)$, then $P', P'', P''', P^{\text{iv}} = (x_1', x_2', x_3', x_4')$, (x_2', x_1', x_4', x_3'), (x_3', x_4', x_1', x_2'), (x_4', x_3', x_2', x_1'), where $x_i' \equiv s - x_i$ and $2s = x_1 + x_2 + x_3 + x_4$, we may easily verify that the five theorems are equivalent to the following analytical restatements of them. The m_i denote odd integers $\gtrless 0$, the l_i even integers $\gtreqless 0$, representing $n > 0$ in the forms

$$\sum_{i=1}^{4} m_i^2, \qquad \sum_{i=1}^{4} l_i^2, \qquad m_1^2 + m_2^2 + l_3^2 + l_4^2,$$

and summations are with respect to all l_i, m_i for a given n. The rest of the notation is:

$$2\lambda_1 = l_1 + l_2,\quad 2\lambda_3 = l_3 + l_4,\quad 2\mu_1 = m_1 + m_2, 2\mu_3 = m_3 + m_4,$$
$$\lambda = \lambda_1 + \lambda_3,\quad \mu = \mu_1 + \mu_3,\quad \nu = \mu_1 + \lambda_3,\quad 2\alpha = m_1 - 1 + l_3,$$
$$l_i' = \lambda - l_i,\quad m_i' = \mu - m_i,\quad l_i'' = \nu - l_i,\quad m_i'' = \nu - m_i.$$

Corresponding to theorems (I)–(V) we now have the following:

(I′) $$\Sigma[(-1)^\mu + 1]f(m_1, m_2, m_3, m_4) = \Sigma[(-1)^\mu + 1]f(m_1', m_2', m_3', m_4'),$$
$$\Sigma[(-1)^\mu - 1]f(m_1, m_2, m_3, m_4) = \Sigma[(-1)^\lambda - 1]f(l_1', l_2', l_3', l_4'),$$
$$\Sigma[(-1)^\lambda + 1]f(l_1, l_2, l_3, l_4) = \Sigma[(-1)^\lambda + 1]f(l_1', l_2', l_3', l_4'),$$
$$\Sigma[(-1)^\lambda - 1]f(l_1, l_2, l_3, l_4) = \Sigma[(-1)^\mu - 1]f(m_1', m_2', m_3', m_4').$$

(II′) $$\Sigma[(-1)^{\mu_1}f(m_1, m_2, m_3, m_4) + (-1)^{\lambda_1}f(l_1, l_2, l_3, l_4)] = \Sigma[(-1)^{\mu_1}f(m_1', m_2', m_3', m_4') + (-1)^{\lambda_1}f(l_1', l_2', l_3', l_4')],$$
$$\Sigma[(-1)^{\mu_1}f(m_1, m_2, m_3, m_4) - (-1)^{\lambda_1}f(l_1, l_2, l_3, l_4)] = \Sigma[(-1)^{\mu_3}f(m_1', m_2', m_3', m_4') - (-1)^{\lambda_3}f(l_1', l_2', l_3', l_4')].$$

(III′) $$\Sigma[(-1)^\nu + 1]f(m_1, m_2, l_3, l_4) = \Sigma[(-1)^\nu + 1]f(l_3'', l_4'', m_1'', m_2''),$$
$$\Sigma[(-1)^\nu - 1]f(m_1, m_2, l_3, l_4) = \Sigma[(-1)^\nu - 1]f(m_1'', m_2'', l_3'', l_4'').$$

(IV′) $$\Sigma[(-)^{\mu_1} + (-1)^{\lambda_3}]f(m_1, m_2, l_3, l_4) = \Sigma[(-1)^{\mu_1} + (-1)^{\lambda_3}]f(l_3'', l_4'', m_1'', m_2''),$$
$$\Sigma[(-1)^{\mu_1} - (-1)^{\lambda_3}]f(m_1, m_2, l_3, l_4) = \Sigma[(-1)^{\mu_1} - (-1)^{\lambda_3}]f(m_1'', m_2'', l_3'', l_4'').$$

(V′) $$2\Sigma(-1)^\alpha g(m_1, m_2, l_3, l_4) = \Sigma(-1)^\alpha[g(l_4', l_3', m_2', m_1') + g(l_3', l_4', m_1', m_2') + g(m_2', m_1', l_4', l_3') - g(m_1', m_2', l_3', l_4')].$$

To pass to the theta functions, replace $f(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $g(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ by the cos, sin respectively of $(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4)$, where x_1, x_2, x_3, x_4 are parameters. In this form the trigonometric sums involving unaccented letters in I′–IV′, and the left of the identity in V′, are readily seen to be the coefficients of q^{4n} or q^{4n+2} in products of the form

$$\vartheta_\alpha(x_1, q^4)\vartheta_\beta(x_2, q^4)\vartheta_\gamma(x_3, q^4)\vartheta_\delta(x_4, q^4),$$

where $\alpha, \beta, \gamma, \delta$ are certain of the numbers 0, 1, 2, 3. Again, the sums involving accented letters in the trigonometric forms of I′–IV′ and the right of V′ are likewise seen to be the coefficients of the same powers of q collected from four such theta products, taken with appropriate signs, in which the variables are $x_i' \equiv s - x_i$, where $2s = x_1 + x_2 + x_3 + x_4$. In this way we find I′–V′ in their trigonometric cases to imply eleven theta identities, which, as they are easily accessible in H. J. S. Smith's second paper on the multiplication formula of four theta functions (Papers, volume 2, page 279), we need not transcribe. It will be sufficient to state the particular formulas of Smith which the trigonometric forms of I′–V′ thus imply. The results in I′ give Smith's (i) $\pm$ (ii), (iii) $\pm$ (iv), and therefore (i) – (iv) in his set A. Similarly his B, C are implied by our II′, III′; his (ix), (xi) of D by our IV′, and his (x) by our V′. From these eleven independent theta formulas, the theory of the theta and elliptic functions, as is well known, follows readily; in fact Smith so derives the theory in his "Memoir on the theta and omega functions" (Papers, volume 2, page 415). The analogous derivation by Jacobi in a famous memoir (Werke, volume 1, page 499) differs only in details. His set (A) is the equivalent of Smith's A–D. We have followed Smith's development rather than Jacobi's because it is the more symmetrical. Jacobi's gives another geometrical image of the theory, and Kronecker's well known exposition of Jacobi's methods yet a third, in which the simpler regular solids inscribed in S play an interesting part.

Now conversely I′–V′, and therefore I–V are implied by the eleven theta formulas of Smith. This follows immediately from the method of paraphrase* outlined in this BULLETIN, volume 26, page 220, § 13. Hence I–V imply, and are implied by, the theory of the elliptic theta functions; viz., the two are formally equivalent.

* The proofs of the method are given in "Arithmetical paraphrases," Part I, to appear shortly in the *Transactions*

5. There is an analogous image for the theory of the theta functions of r variables. In it the lattice space is of $2r(r+1)$ dimensions, and the appropriate configurations lie on a system of r four-dimensional spheres immersed in the higher space. For $r > 1$ the image is not expressible in terms of reflexions alone.

THE UNIVERSITY OF WASHINGTON.

NOTE ON THE MEDIAN OF A SET OF NUMBERS.

BY PROFESSOR DUNHAM JACKSON.

(Read before the American Mathematical Society September 7, 1920.)

LET $a_1, a_2, \cdots, a_n$ be a set of real numbers, which may or may not be all distinct. Let

$$S_2(x) = \sum_{i=1}^{n} (x - a_i)^2.$$

The value of x which reduces $S_2(x)$ to a minimum is the arithmetical mean of the numbers $a_1, \cdots, a_n$. If the condition that $S_2(x)$ be a minimum is replaced by the condition that

$$S_1(x) = \sum_{i=1}^{n} |x - a_i|$$

be reduced to a minimum, the median of the a's is obtained. It is uniquely defined whenever n is odd; if the numbers a_i are arranged in order of magnitude, so that

$$a_1 \leqq a_2 \leqq \cdots \leqq a_n,$$

and if $n = 2k - 1$, the median is simply a_k, the middle one of the a's. The median is uniquely defined also when n is even, $n = 2k$, if it happens that $a_k = a_{k+1}$, being then equal to this common value. Otherwise, the definition is satisfied by any number x belonging to the interval

$$a_k \leqq x \leqq a_{k+1},$$

and the median is to this extent indeterminate.

The purpose of the following paragraphs is to show that