THE MATHEMATICAL WORK OF THOMAS JAN STIELTJES.

Œuvres complètes de Thomas Jan Stieltjes. Publiées par les soins de la Société Mathématique d'Amsterdam. P. Noordhoff, Groningen. Tome I, 1914: viii + 471 pp. Tome II, 1918: iv + 604 pp.

ONE can not assert that Thomas Jan Stieltjes was one of the great men of the earth. In fact he was not one of the greatest men in the narrower circle of his colleagues in mathematical investigation. But he was a man of fine talent who used the full strength of his powers in his researches; and in his short life (1856–1894) he did excellent work which deserves to be remembered. It is therefore fitting that his articles and memoirs should be brought together in the convenient form of a collected edition. In the two volumes of this work and in the two volumes which record the correspondence between Stieltjes and Hermite we have a complete record* of the scientific activity of Stieltjes and a clear and pleasing insight into his methods of work. The latter is more apparent in the letters, the less formal nature of which gave rise to a more intimate revelation of himself to his friend and later to the world.

His earlier work especially is marked by a careful and deep study of particular questions and the interest which he took in adapting algebraic and analytic formulas to numerical computation. This seems to have been due in considerable measure to the fact that he approached mathematics from the point of view of one engaged in astronomical work and only ater gave up what was first conceived to be the object of his scientific life in favor of his studies in mathematics which during the years 1877 to 1883 gained a stronger and stronger grasp upon his thought.

Many of his discoveries were made empirically by incomplete induction from numerous examples developed from the beginning with the delight which he always evinced in numerical computation. In this respect his method has been compared

* See also the section devoted to the (unpublished) "method of Stieltjes" (pp. 357–362, 323) in Poincaré's memoir on the residues of double integrals in *Acta Mathematica*, volume 9, 1887.

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with that of Gauss who is known to have discovered experimentally many of his beautiful theorems in the theory of numbers. This method has a peculiar power when it is employed by an intellect of sufficiently keen penetration to divine the general law in the midst of the special properties which belong to particular examples. The procedure of discovery in the case of Stieltjes seems to have been dominated by this empirical method. It seems to be true that nearly all the mathematical truths which he made known were discovered in this way before he was in possession of methods of proof; and that the latter were obtained afterwards by a penetrating analysis of the essential elements of his problem. The truths thus revealed by experiment were subjected to the acid test of logical demonstration and he was mostly free from the enunciation of results for which he had no adequate proof. His letters show the extreme care which he took in the matter of rigorous argumentation. His conversation is said to have shown a wide acquaintance with what may be called interesting mathematical phenomena which his patient calculations had brought to light but which he knew only partially through empirical evidence and not with the clarity and certainty and accuracy of demonstrated truth.

But he seems to have fallen at least once from the high plane of logical precision and accuracy which his published results usually occupied and to have stated a theorem for which he could give no satisfactory proof (though the theorem is probably true). He was engaged (Volume I, page 457) in an investigation of Riemann's celebrated conjecture concerning the distribution of the zeros of the Riemann zeta-function. He transformed the problem so that the truth of the theorem would follow from the convergence of a certain Dirichlet series whose coefficients depended on the function-values of a certain number-theoretic function $g(n)$. This function he examined for a certain property through the ranges of n from 1 to 1200, from 2000 to 2100, and from 6000 to 6100; and, finding the property maintained in these regions for n, he concluded that it was a universal property of $g(n)$. The proof by which he first satisfied himself logically of the existence of the property (see letter 79) he seems never to have published; and the inference to be drawn is that he later found it lacking in some point of accuracy or rigor.

One who reviews the work of Stieltjes at the present time

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is spared the necessity of making an analysis of his separate memoirs, for this was done by E. Cosserat soon after the death of Stieltjes in a "Notice" of 62 pages appearing as the opening article of volume 9 (year 1895) of the *Toulouse Annales*. Here we have a résumé of each of the 84 papers now appearing in his Œuvres complètes, with the exception of the last three (which are minor contributions made up by his editors from his unpublished manuscripts). The student of the work of Stieltjes will find these abstracts very valuable for making a rapid survey of the extent of his contributions. [The forty-fifth abstract (and hence the forty-sixth) is perhaps misleading since it quotes the results of Stieltjes without reference to his failure to make good that one of his statements just mentioned in our preceding paragraph.] His great paper (number 80 of the Œuvres) on continued fractions is here discussed only briefly. But one interested in analyzing the contributions made by Stieltjes will certainly wish to examine this paper in full for himself, so that he will not suffer from the absence of a fuller review.

Again, a reviewer at the present time is under no obligation to give a sketch of the life of Stieltjes and relate his scientific investigations to what is thought of usually as the more human aspects of one's career, for this has already been done well by H. Bourget in his "Notice sur Stieltjes" in pages xi to xx of the first volume of the correspondence between Stieltjes and Hermite.

There remains then only the duty to call attention to the features of Stieltjes' work which are of outstanding interest or importance when viewed in the light of present knowledge or which are presumably of particular value to those who now carry the torch of science which he upheld so devotedly with all the strength of his fine talent.

For the purposes of discussion it is convenient to divide the active scientific life of Stieltjes into two parts. The first ends at the time when the influence of the Paris environment and labor and training began to be evident in his work, that is, about 1886 (though he went to Paris in April, 1885). The second period falls in the remaining eight years of his life. The physical bulk of his contributions, exclusive of the 113-page expository paper on the theory of numbers (which falls in the second period), was nearly equal in the two periods; but the two parts are of distinctly unequal merit. Whether

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it was the intention of the editors to do so or not I do not know, but they have made the first and second volumes of his Œuvres cover each quite exactly one of these periods in his life, so that the physical division into volumes corresponds closely to a division of his work into distinct parts.

The two parts which may thus be distinguished and separated are yet intimately related and bear throughout the impress of their author's individuality. But the first is given more largely to special problems and shows more clearly the unfolding of the power of Stieltjes' thought, while the latter abounds more in the finished work of his maturer years. Yet in this first volume is to be found the beautiful contribution to the theory of cubic and biquadratic residues, his researches on mechanical quadratures, and his interesting paper on the variation of the density of the earth.

The last three or four years of the first period of his life were particularly full of activity. In this interval he was married, he began to develop what became a life-long intimate friendship with Hermite, and he had the stimulus due to his new abode in Paris. The change and development in the more external affairs of his life were repeated in, or at least had their counterpart in, the remarkable development of his spirit which took place at the same time.

The ingenious conceptions, the generating ideas, the germs of his later activity, multiplied during this interval and in the year or two which followed. To this epoch of his life nearly all his most interesting contributions are to be referred either for their completion or at least for their initial conception. In the few remaining years he developed and extended the ideas and solved some of the problems which had already arisen in his mind.

The centering of the more intensely creative activity of Stieltjes' life so largely in a single short period of it is perhaps instructive. In this period more than in any other new forces moved upon him from without and varied new delights (from new family ties, from the new friendship with Hermite, and from the new opportunity to give all his thought to mathematics) wrought upon his character and life and outlook to make him as it were a new individual. In this period the most of his essentially creative work was done. Is it perhaps true generally that the spirit of man yields its greatest return in new truth discovered when it is subjected to the exhilaration

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of the maximum of pleasant change in environment and outlook and subject of interest? This question first arose in my mind and an affirmative answer pressed itself upon my thought when I was once studying the relation of the greater plays of Shakespeare to the character of his life at the time when they were produced; and I have often had occasion to observe a like correspondence in the external life and the more highly creative periods of numerous thinkers in widely separated fields of activity. The work of Stieltjes, while not that of a great master, seems to me nevertheless to be an instructive case in point. If my thesis is well founded it suggests a question of profound value as to how one shall maintain his own thought at the highest possible level of creative exaltation.

It remains to discuss briefly a few of the more important memoirs of the second period in Stieltjes' scientific life. The first of these is his Paris doctor's thesis on certain semi-convergent series. In his work as an astronomer he had often observed the usefulness of certain divergent series (analogous to the celebrated formula of Stirling in the theory of the gamma function) for the purposes of numerical computation and he realized that the theory of these series had never been put on a satisfactory basis. He set himself the task to make a systematic analysis of the matter so that at the end of his study he might be justified in looking upon the series as the asymptotic representation of one or more functions. He studied carefully certain divergent series of special importance and developed their properties so as to obtain from them the most exact information possible as to the numerical values of functions associated with them. He was proceeding largely by his empirical method of gradual approach to the central facts of importance and wide generality; and, if left to himself in this study, he would probably have pushed the investigation to a much wider range. But the genius of Poincaré had turned almost simultaneously and quite independently to this same problem which astronomy had dumbly set before the mathematician for so long a time, and he attacked the problem from a general point of view rather than through special cases; under the fire of his genius the leading secrets were brought to light. Doubtless much remains to be done with the general problem, but the work of Poincaré so overshadowed the simultaneous contribution of Stieltjes that the latter did not pursue the question in further extensive researches; and

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his interesting and worthy contribution has largely fallen out of notice.

The most significant among the works of Stieltjes is his celebrated memoir entitled "Recherches sur les fractions continues" (printed on pages 402 to 566 of the second volume of the Œuvres). Here several partial investigations in his earlier work reach their full stature in completed theorems. The germs of some of the most fruitful ideas in the memoir go back as far as the later active years of the first period of his scientific life; and several partial results from his previous scientific activity are woven into it in their proper place and brought to a further stage of completeness than in the earlier papers. In volume 119 of the Paris *Comptes rendus* (at pages 630–632) one will find a report by Poincaré on this memoir. It is described as one of the most remarkable memoirs in analysis "written in the last years," and it is said to place its author in an eminent rank "in the Science of our epoch." In the "Notice" of Bourget we have an interesting, and even a touching, account of the way in which the last discoveries of Stieltjes, which made it possible for him to bring this memoir to completion, so fired the zeal of his spirit for several months as to keep his mind in the freshness of its power even though his last illness was already sapping his physical strength and bringing him to a state of weakness in which he was unable to take up any further labors. He died a few months after the completion of the memoir.

The reputation of Stieltjes can never go higher than the researches recorded in this memoir can take it; and it can never fall lower than the level to which this memoir would bring it; for, though several of his papers contain nothing more than the solutions of problems not inherently difficult, our judgment of the quality of his work will be based primarily on the character of his most worthy effort. In this memoir we find all the qualities of elegance and clarity and marked originality which are characteristic of his better work.

The continued fractions considered by Stieltjes are such that the incomplete quotients are alternately of the forms $a_{2n+1}z$ and a_{2n}, the a_i being real and positive, so that the fraction may be written in the form

$$\frac{1}{a_1 z} + \frac{1}{a_2} + \frac{1}{a_3 z} + \frac{1}{a_4} + \cdots.$$

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His central result may be stated as follows: If the series Σa_n converges the fraction is oscillatory; the approximating reduced fractions of even order tend to one limit and those of odd order tend to another limit; in each of these two sequences of fractions the numerator and the denominator tend each to an integral function of "genre" zero all of whose roots are real and negative; the limiting form of each of these two sequences from the approximating reduced fractions is a function which is meromorphic throughout the finite plane and is decomposable into a series of simple partial fractions. If the series Σa_n diverges, the continued fraction is convergent and the limit is a function $F(x)$ which is holomorphic everywhere except (possibly) on the negative axis of reals; this function $F(x)$ can be represented by a certain definite integral which in certain cases may be replaced by a series of simple fractions. In connection with the proof of these results Stieltjes derives and utilizes (Chapter V) a remarkable theorem in the general theory of functions, treats (Chapter VI) his celebrated problem of moments and defines (Chapter VI) the now classic Stieltjes integral. An extension of this memoir of Stieltjes has been given by Van Vleck (*Transactions of the American Mathematical Society*, volume 4 (1903), pages 297–332). See also Van Vleck's Boston Colloquium lectures, pages 147–152.

Probably Stieltjes will be longest and most gratefully remembered for his introduction of the integral now called after his name. He employed it in 1894 incidentally to the solution of his continued fractions problem and did not undertake to develop its properties further than was needful for such a use of it. The great importance of the new limiting process was not at once realized and the possibility of its use remained latent for a number of years. But it is more recently coming into its own. Its place is made abundantly clear by Hildebrandt in this BULLETIN, volume 24, 1918, pages 178 ff., in his excellent statement of the reasons why it must be considered a conception of fundamental importance. Perhaps the two strongest are those which arise from the following two results: for every linear functional operation $U(f)$ on continuous functions $f(x)$ there exists a function $u(x)$ of bounded variation such that

$$U(f) = \int_a^b f(x)du(x),$$

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the integral being taken in the sense of the definition of Stieltjes; a necessary and sufficient condition that every continuous function on (ab) may be uniformly approximated by linear combinations of a set of functions $[\varphi_1(x), \cdots, \varphi_n(x), \cdots]$ is that the only solution of the equations

$$\int_a^b \varphi_n(x) du(x) = 0, \qquad (n = 1, 2, \cdots)$$

for a $u(x)$ which for every x_0 shall satisfy the condition

$$u(x_0) = \tfrac{1}{2}[u(x_0 + 0) + u(x_0 - 0)]$$

is $u(x) =$ constant.

The way in which Stieltjes came to the introduction of his new limiting process is interesting as illustrating one extreme of the method of discovery. It is as far removed as possible from that method in which one sets out deliberately to extend or generalize conceptions previously found interesting or useful in order to ascertain what further things of value may be seen to grow out from them. Such a method a man of Stieltjes' scientific temperament could never have employed. His mind did not operate in the direction of extending known conceptions by meditating upon them; and apparently he could never have succeeded in working in this way, however well the method may be suited to a certain different and perhaps more robust scientific temperament. On the contrary he proceeded first with particular instances of the problem of a general investigation and solved a number of relatively simple special problems which arose in this way. Being cast down repeatedly by certain difficulties which he could not at first surmount and analyzing the tools which he had employed to ascertain why they did not carry him to the goal, he seems to have come to a realization that the integral which he had been using was not altogether as flexible as the exigencies of his problem demanded. Certain aspects of it he was able to look upon dynamically as a problem in moments. But the moment could not be expressed always in the form of an ordinary integral though it was certainly the limit (in a certain sense) of a sum much like that employed in the Cauchy-Riemann definition of integral. This limit afforded him his new integral. In this way arose one of the highly fruitful conceptions of recent mathematical analysis—one

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which will probably play a role of fundamental importance in the further development of certain central branches of mathematics.

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SHORTER NOTICES.

Opere di Evangelista Torricelli. Edited by GINO LORIA and GIUSSEPPE VASSURA. Faenza, 1919, 2 volumes. Volume I, part 1, xxxviii + 408 pp.; part 2, iv + 482 pp. + plates; Volume II, iv + 322 pp. + plates.

OF those who sat at the feet of Galileo (1564–1642) and from him received instruction and inspiration, two were permitted to enjoy this privilege only in the last weeks of his life. One of these, Viviani (1622–1703), was fifty-eight years his junior and was only twenty years of age when the great teacher passed away. Viviani survived Galileo by sixty-one years, the lives of the two bridging a span of nearly a century and a half. With propriety as well as with pride he could say, in his later life, that he was "postremus Galilei discipulus." In a way, however, Torricelli (1608–1647) could have said the same, for he too was one of the last of those who learned from the great master, although he died so early that he was not, like Viviani, the last disciple to pass away. Viviani signed his famous problem on the hemispherical dome by an anagram of the words "A postremo Galilei Discipulo," while Torricelli was proud to observe that the letters of his own name could be transposed to form the sentence "En virescit Galileus alter."

Of these two great disciples the more brilliant was Torricelli. With a span of life that was less than half as long as that of Viviani, he may be said to have accomplished twice as much, and the results of his labors have been set forth in the volumes under review.

Volume I, consisting of two parts, covers the work of Torricelli in the field of geometry and appears under the editorship of Professor Loria, while Volume II includes his academic lectures, his work in mechanics, and his writings in various minor lines, and is published "per cura" of Professor Vassura.

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