ON THE FOURIER COEFFICIENTS OF A CONTINUOUS FUNCTION.

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It is well known that when

$$\frac{a_0}{2} + \sum_{1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$$

is the Fourier expansion of a function $f(\theta)$ which is real and continuous for $0 \leq \theta \leq 2\pi$, then $\Sigma(a_n^2 + b_n^2)$ converges. Here the exponent 2 cannot in general be replaced by a smaller one; in fact, Carleman* has constructed an example of a continuous $f(\theta)$ where $\Sigma(a_n^{2-\delta} + b_n^{2-\delta})$ diverges for any $\delta > 0$, and this example has been simplified by Landau, †

In the present note it will be shown that, given any single-valued real function $\varphi(x)$, subject only to the condition that $\varphi(x)$ becomes infinite as $x$ becomes infinite, there exists a real continuous function $f'(\theta)$ whose Fourier coefficients $a_n, b_n$ make the series

$$\sum (a_n^2 + b_n^2)^{\delta} \varphi \left( \frac{1}{a_n^2 + b_n^2} \right)$$

divergent. Assuming $\varphi(x) = x^\delta$, where $\delta > 0$, and observing that $(a^2 + b^2)^{1-\delta} < a^{2-\delta} + b^{2-\delta}$, we have the particular result referred to above.

If we denote by $f_1(\theta)$ the function conjugate to $f(\theta)$, and write $z = e^{i\theta}$, $F(z) = f(\theta) + if_1(\theta)$, the Fourier expansion of $F(z)$ is

$$\sum_0 \varphi \left( \frac{1}{c_n^2} \right) c_n z^n,$$

where $c_0 = a_0/2$, $c_n = a_n - ib_n$ ($n > 0$).

Our statement will be proved by constructing a function $F(z)$ continuous for $|z| = 1$ and such that $\Sigma |c_n|^2 \varphi(1/|c_n|^2)$ diverges. This will be done by means of the following result due to Hardy and Littlewood‡ and used by Landau, loc. cit., for a different purpose:

Let \( \xi \) be a real irrational number such that all the denominators in its expansion in a continued fraction are bounded (for instance \( \xi = \sqrt{2} \) or any quadratic irrationality). Then there exists an \( A = A(\xi) \) independent of \( n \) and \( \xi \) such that for any \( n \geq 1 \), and any \( \xi \) on the unit circle \( |\xi| = 1 \),

\[
\sum_{\nu=1}^{n} e^{i2\pi \xi \nu^2} |z^\nu| < A \sqrt{n}.
\]

Making

\[
F_\nu(z) = \sum_{\nu=1}^{n} e^{i2\pi \xi \nu^2 / \sqrt{n}} z^{\nu}
\]

we have therefore \( |F_\nu(z)| < A \) for \( |\xi| = 1 \); writing \( k_\nu = n_0 + n_1 + \cdots + n_{\nu-1} \) and assuming \( d_\nu \) to be such that \( \Sigma |d_\nu| \) converges, we find that the series

\[
F(z) = \sum_{\nu=0}^{\infty} d_\nu z^{k_\nu} F_\nu(z)
\]

converges uniformly for \( |\xi| = 1 \), so that \( F(z) \) is continuous on the unit circle. Multiplying by \( z^{-n-1} dz \) and integrating along the unit circle, we may integrate term by term to the right on account of the uniform convergence, and the Fourier coefficients \( c_n \) of \( F(z) \) are thus found to be

\[
c_n = d_\nu \frac{e^{i\xi \nu^2}}{\sqrt{n}} \quad (n = k_\nu + 1, k_\nu + 2, \cdots, k_\nu + n_\nu).
\]

Consequently

\[
\sum_{n=k_\nu+1}^{k_\nu+1} |c_n|^2 \varphi \left( \frac{1}{|c_n|^2} \right) = d_\nu \varphi \left( \frac{n}{|d_\nu|^2} \right),
\]

and since \( \varphi(x) \) becomes infinite as \( x \) becomes infinite, we may choose each \( n_\nu \) so that \( \varphi \left( \frac{n}{|d_\nu|^2} \right) \) diverges, which proves our theorem.

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