cases being that this value may actually be attained in the latter case for one or more special values of θ.*

Finally, it may be observed that the values of the so-called absolute minima for the cases where area may be passed over four, five, six, $\cdots$ times are respectively $\frac{1}{4}l^2\theta$, $\frac{1}{5}l^2\theta$, $\frac{1}{6}l^2\theta$, $\cdots$. The consideration of these cases, however, on the geometrical side again presents serious difficulties, but tends to the opinion, as in the case of triplication, that *in general* the smallest area that can be swept over by any actual movement of angle θ is $\frac{1}{2}l^2\theta$ rather than any of these smaller values.

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CONVERGENCE OF SEQUENCES OF LINEAR OPERATIONS †

BY T. H. HILDEBRANDT.

Let U_n be a sequence of linear continuous operations on the class F of functions f, continuous on the interval (a, b), i.e., suppose that every U satisfies the two conditions:

$$(1) \qquad U(c_1f_1 + c_2f_2) = c_1U(f_1) + c_2U(f_2)$$

for every pair of constants (c_1, c_2) and every pair of functions (f_1, f_2) of the class F;

(2) There exists a constant M depending on U such that if Nf is the maximum value of $|f|$ on (a, b) then

$$|U(f)| \leqq MNf.$$

The greatest lower bound of all possible values M might be called the *modulus* of U.

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* Thus, in case $\theta = \pi$ and triplication is allowed, the corresponding value $\frac{1}{3}l^2\pi$ may be attained as follows: Construct the hypocycloid of three cusps obtained by rolling the circle of radius $\frac{1}{2}l$ within the circle of radius $\frac{3}{2}l$ and let the given segment (of length $2l$) move so as to be always tangent to this curve and yet be everywhere entirely within it. The resulting area swept over as θ passes from 0 to π is entirely triplicated, as is well known, and is equal to the amount above stated, $\frac{1}{2}l^2\pi$. See, for example, F. Gomes Teixeira, *Traité des Courbes Spéciales Remarquables Planes et Gauches*, vol. II, p. 193. (Coïmbre, 1909.)

† Presented to the Society, September 4, 1919.

Helly* has shown that a necessary condition that $\lim_n U_n$ exists for every f of F is that the U_n be uniformly bounded, i.e., that there exist a constant M independent of n such that

$$|U_n(f)| \leqq MNf.$$

Then

$$\lim_n U_n(f) = U(f),$$

where $U(f)$ is a linear continuous operation whose modulus is less than or equal to M. In order to obtain sufficient conditions, it is necessary to make use of the classical theorem of Riesz, that every linear continuous operation on the class F is expressible in the form of the Stieltjes integral, $\int f d\alpha$, where α is of bounded variation. If α is regular, i.e., such that at every point x on (a, b) $\alpha(x)$ lies between $\alpha(x-0)$ and $\alpha(x+0)$, which can always be assumed to be the case without changing the value of the integral (and we shall restrict ourselves to this case), then the total variation ($\int|d\alpha|$) of α is exactly the modulus of U, so that Helly's condition would be, that there exists a constant M independent of n such that

$$\int|d\alpha_n| < M,$$

for every n, i.e., the α_n are *uniformly of bounded variation.*

This condition is not sufficient. Helly† has shown that if (1) the α_n are uniformly of bounded variation and (2) there exists a function α of bounded variation such that

$$\lim_n \alpha_n = \alpha \text{ for every } x,$$

then

$$\lim_n \int f d\alpha_n = \int f d\alpha \text{ for every } f \text{ of } F.$$

Bray‡ has shown that a weaker second condition is that there exists a function of bounded variation α and a denumerable everywhere dense set of points: $x_1, \cdots, x_m, \cdots$ including a and b, such that for every x_m

$$\lim_n \alpha_n(x_m) = \alpha(x_m).$$

This condition is not necessary. It is the purpose of this note to derive necessary and sufficient conditions.

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* Helly, WIENER SITZUNGSBERICHTE, vol. 121 (IIa) (1912), p. 268.

† Loc. cit., p. 288.

‡ ANNALS OF MATHEMATICS, (2), vol. 20 (1919), p. 180.

We derive first an additional necessary condition. Since the α_n are necessarily uniformly of bounded variation, it follows from a theorem by Helly* that the α_n are a compact set, i.e., there exists a subsequence α_{n_m} of the sequence α_n and a function α (necessarily of bounded variation) such that

$$\lim_m \alpha_{n_m} = \alpha \text{ for every } x.$$

Consequently

$$\lim_m \int f d\alpha_{n_m} = \int f d\alpha$$

for every f of F. Since $\lim_n \int f d\alpha_n$ exists, it follows that

$$\lim_n \int f d\alpha_n = \int f d\alpha, \qquad \text{or} \qquad \lim_n \int f d(\alpha_n - \alpha) = 0$$

for every f of F. Let

$$\beta_n(x) = \alpha_n(x) - \alpha(x) - \alpha_n(a) + \alpha(a).$$

Then our condition becomes:

$$\lim_n \int f d\beta_n = 0$$

for every f of F. If we take $f = 1$ then

$$(a) \qquad \lim_n [\beta_n(b) - \beta_n(a)] = \lim_n \beta_n(b) = 0.$$

Again take $f = x$ for $a \leqq x \leqq \xi$ and $f = \xi$ for $\xi \leqq x \leqq b$. Then if we apply the integration by parts formula valid for Stieltjes integrals:

$$\int f d\beta_n = f\beta_n \Big|_a^b - \int \beta_n df = \xi\beta_n(b) - \int_a^\xi \beta_n dx.$$

Consequently, since $\lim_n \beta_n(b) = 0$, we must have

$$(b) \qquad \lim_n \int_a^x \beta_n dx = 0 \text{ for every } x \text{ of } (a, b)$$

or the equivalent condition:

$$(b') \quad \lim_n \int_{a_1}^{b_1} \beta_n dx = 0 \text{ for every subinterval } (a_1, b_1) \text{ of } (a, b).$$

We transform this last condition as follows:

LEMMA I. *If β_n is a sequence of functions uniformly of bounded variation such that*

$$\lim_n \int_{a_1}^{b_1} \beta_n dx = 0$$

for every subinterval (a_1, b_1) of (a, b), and if u_n is the greatest lower bound of $|\beta_n(x)|$ for x on (a, b), then $\lim_n u_n = 0$.

* Loc. cit., p. 283. See also Radon: WIENER SITZUNGSBERICHTE, vol. 122 (IIa), p. 1377, and Fischer, this BULLETIN, vol. 27 (1920), p. 12.

If $\lim_n u_n$ is not zero, then there will exist an $e > 0$ and a subsequence u_{n_m} of u_n such that $u_{n_m} > e$ for every n_m, i.e., for every n_m and every x we have

$$|\beta_{n_m}(x)| > e.$$

Since the β_{n_m} form a compact set, there exists a subsequence β_k' approaching a limiting function β also of bounded variation. This function β will obviously be such that $|\beta(x)| \geqq e$ for every x of (a, b). Consequently there exists a subinterval (a_1, b_1) of (a, b) such that either

$$\beta(x) \leqq -e \qquad \text{or} \qquad \beta(x) \geqq e$$

for every x of (a_1, b_1), i.e.,

$$\left|\int_{a_1}^{b_1} \beta(x)dx\right| > e(b_1 - a_1).$$

But the β_k' are uniformly bounded. Hence*

$$\lim_k \int_{a_1}^{b_1} \beta_k'(x)dx = \int_{a_1}^{b_1}\beta(x)dx,$$

which is not zero. Then we have reached a contradiction to the hypothesis of the lemma.

A direct consequence of this lemma is:

LEMMA II. *If β_n is a set of functions uniformly of bounded variation such that*

$$\lim_n \int_{a_1}^{b_1}\beta_n(x)dx = 0$$

for every subinterval (a_1, b_1) of (a, b), then in every subinterval (a_1, b_1) of (a, b) there exists a sequence of points x_n such that

$$\lim_n \beta_n(x_n) = 0.$$

For, any subinterval (a_1, b_1) of (a, b) may replace (a, b) in the hypothesis of Lemma I. Moreover, since $\lim_n u_n = 0$, it follows that there will exist a point x_n such that

$$|\beta_n(x_n)| \leqq u_n + \frac{1}{n},$$

i.e., $\lim_n \beta_n(x_n) = 0$.

The conclusion of this lemma together with the fact that the α_n are uniformly of bounded variation is also sufficient for the convergence under consideration, i.e., we have:

* Cf. Lebesgue, *Leçons sur l'Intégration*, p. 114.

THEOREM. *Necessary and sufficient conditions that the limit* $\lim_n \int_a^b f d\alpha_n$ *exist for every* f *of* F *are that*

(1) *the* α_n *be uniformly of bounded variation,*

(2) *there exist a function* α *of bounded variation such that if*

$$\beta_n(x) = \alpha_n(x) - \alpha_n(a) - \alpha(x) + \alpha(a),$$

then

(*a*) $\lim_n \beta_n(b) = 0,$

(*b*) *in every subinterval* (a_1, b_1) *of* (a, b) *there exists a set of points* $x_1, \cdots x_n, \cdots$ *such that*

$$\lim_n \beta_n(x_n) = 0.$$

The function α of the theorem may be taken to be the function which is the limit of a subsequence of $\alpha_n(x)$.

To prove the sufficiency, we show that, under the hypotheses of the theorem, $\lim_n \int f d\beta_n = 0$ for every function f of F. From a theorem of Bray* it follows that

$$\left|\int f d\beta_n - \sum_{i=1}^{m} f(\xi_i)[\beta_n(x_{i-1}) - \beta_n(x_i)]\right| \leqq O_\delta M,$$

where $x_0 = a, x_1, \cdots, x_{m-1}, x_m = b$ is a subdivision of (a, b), ξ_i lies in the interval (x_{i-1}, x_i), O_δ is the maximum oscillation of $f(x)$ in (x_{i-1}, x_i) and $\int |d\beta_n| \leqq M$ for every n. Since f is uniformly continuous in (a, b), for every e, there will exist a d_e such that if $|x_i - x_{i-1}| \leqq d_e$, then $O_\delta \leqq e/2M$. Take a subdivision of (a, b) by the points $\xi_1 = a, \xi_2, \cdots, \xi_m = b$ such that for every i, $|\xi_i - \xi_{i-1}| \leqq \frac{1}{2} d_e$. Let $4d_0$ be less than the minimum of $\xi_i - \xi_{i-1}$. We apply the hypothesis of the theorem to the intervals $(\xi_i + d_0, \xi_{i+1} - d_0)$, i.e., select a set of points $x_{i,n}$ in each interval such that $\lim_n \beta_n(x_{i,n}) = 0$. We take $x_{0,n} = a, x_{1,n}, \cdots, x_{m,n} = b$, as points of division. Then since these points are finite in number, we can find an n_e such that if $n \geqq n_e$

$$\left|\sum_{i=1}^{m} f(\xi_i)[\beta_n(x_{i,n}) - \beta_n(x_{i-1,n})]\right| \leqq e/2.$$

* Loc. cit., p. 179.

Since we also have

$$\left| \int f d\beta_n - \sum_{i=1}^{m} f(\xi_i)[\beta_n(x_{i,n}) - \beta_n(x_{i-1,n})] \right| \leqq e/2,$$

we have for $n \geqq n_e$

$$|\int f d\beta_n| \leqq e \qquad \text{or} \qquad \lim_n \int f d\beta_n = 0.$$

The conditions of the theorem may be simplified, if we note that the converse of Lemma II holds in the following form.

LEMMA II′. *If β_n is any set of functions uniformly of bounded variation, then a sufficient condition that* $\lim_n \int_a^x \beta_n dx = 0$ *for every x of (a, b) is that in every subinterval (a_1, b_1) of (a, b) there exist a sequence of points $x_1, \cdots, x_n, \cdots$ such that* $\lim_n \beta_n(x_n) = 0$.

The proof of this can be made along the lines of the sufficiency proof above.

As a consequence our theorem may be stated as follows.

Necessary and sufficient conditions that $\lim_n \int f d\alpha_n$ *exist for every f of F are that:*

(1) *the α_n be uniformly of bounded variation;*

(2) *there exist a function α of bounded variation such that if $\beta_n(x) = \alpha_n(x) - \alpha_n(a) - \alpha(x) + \alpha(a)$, then*

$$\lim_n \beta_n(b) = 0, \qquad \textit{and} \qquad \lim_n \int_a^x \beta_n dx = 0 \textit{ for every } x.$$

In this form it contains the following theorem of Lebesgue* as a special case. Necessary and sufficient conditions that $\lim_n \int f\varphi_n = 0$ for every f of F, the φ_n being summable on (a, b), are:

(1) $\int |\varphi_n| dx$ be bounded as to n;

(2) (*a*) $\lim_n \int_a^b \varphi_n \, dx = 0$;

(*b*) for every y, $\lim_n \int_y^b (y - x)\varphi_n(x) dx = 0$.

This results from the above theorem if we put

$$\beta_n(x) = \int_a^x \varphi_n(x) dx.$$

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* *Sur les intégrales singuliers*, ANNALES DE TOULOUSE, (3), vol. 1 (1909), p. 57.