A TWO-WAY INFINITE SERIES FOR LEBESGUE INTEGRALS*

BY M. B. PORTER

1. *The Defining Series.* Let $F(x)$ denote any positive measurable function of n variables defined over a measurable point set δ. We shall use the sign δ to denote either a point set or its measure, and the sign† δ^* to denote δ diminished by a nul-set or to indicate that a property holds *almost everywhere* (*presque partout*) over δ.

Now write

$$(1) \qquad \Phi(\delta) = \sum_{i \to -\infty}^{\infty} 2^i \phi_i(\delta) = \operatorname{Lim}_{m \to \infty} \sum_{i \to -\infty}^{m} 2^i \phi_i(\delta),$$

where $\phi_i(\delta)$ is the measure of all those points E_i of δ for which $F(x)$, expressed in the binary scale, has unity in the ith place, it being understood that $F(x)$ shall be always expressed in closed form when possible (or else never so expressed), in order that the representation shall be unique. If $F(x)$ is limited, it is evident that (1) will converge uniformly over δ. If $F(x)$ is *not* limited, (1) may still converge; if it does, the convergence is necessarily uniform. In that case $F(x)$ is summable over δ, and we shall indicate the sum of the series by $\int F(x)$, the Lebesgue integral of F over δ. The integral is definite or indefinite according as we regard δ as fixed or as variable.

In case $F(x)$ is not always positive, set, as usual,

$$\phi(x) = F(x) \text{ when } F(x) \geqq 0, \quad \psi(x) = F(x) \text{ when } F(x) < 0,$$
$$\phi(x) = 0 \text{ when } F(x) < 0, \quad \psi(x) = 0 \text{ when } F(x) \geqq 0;$$

then

$$(2) \qquad \Phi(\delta) = \sum_{-\infty}^{\infty} 2^i \phi_i(\delta) - \sum_{-\infty}^{\infty} 2^j \psi_j(\delta),$$

and this difference may converge even when the separate series diverge, if i and j become simultaneously infinite in a suitable manner.

* Presented to the Society, September 9, 1921.

† Cf. Pierpont's useful notation, *Functions of a Real Variable*, vol. 1, p. 119.

Thus, if

$$F(x) = \frac{d}{dx}\left(x \sin \frac{1}{x}\right), \qquad x \neq 0,$$

$F(x)$ is not summable at $x = 0$; but if from the interval (a, x), $a < 0$, $x > 0$, we remove the interval 2ϵ symmetric about the origin, the integral (2) will exist over the remaining domain and will be equal to $x \sin (1/x) - a \sin (1/a)$. A similar procedure can be applied where $F(x)$ has a countable set of non-summable points, and is the derivative of a continuous function.

Let η denote a set of infinitesimal hyperspheres about x, a point of δ, as center. We shall show that

$$\lim_{\eta \to 0} \frac{\Phi(\eta)}{\eta} \tag{3}$$

exists over δ^*. Since this is a point-function we shall use the argument x in the limit.

Lebesgue has shown in his metric-density theorem that for the measure functions ϕ_i the limit (3) exists almost everywhere, and that†

$$\begin{aligned} \phi_i'(x) &= 1 \quad \text{over} \quad E_i^*, \\ \phi_i'(x) &= 0 \quad \text{over} \quad C^*E_i, \end{aligned} \tag{4}$$

which in one dimension implies the existence of the derivative in the ordinary sense of the word over E_i^* and $C_i^*E_i$ with the values given in (4).

2. *Special Theorems.* From (4) we have the following theorems.

THEOREM A. *For any measurable function $F(x)$,*

$$F'(x) = \sum_{-\infty}^{\infty} 2^i \phi_i'(x) - \sum_{-\infty}^{\infty} 2^j \psi_j'(x),$$

both series converging over that part of δ^* where $F(x)$ is finite.

THEOREM B. *In case $F(x)$ is limited, $\int_\delta F(x)$ exists and its derivative is*

$$\int_\delta{}' F(x) = \sum_{-\infty}^{m} 2^i \phi'(x) - \sum_{-\infty}^{m} 2^j \psi'(x) = F(x)$$

over δ^*, the series converging uniformly over δ.

† Various proofs of this theorem have been given, e.g. a simple proof based on Vitali's covering theorem by de la Vallée Poussin, *Cours d'Analyse*, vol. 2, p. 110 et seq. See also Hobson, *Real Variables*, vol. 2, p. 178 et seq.

If $\psi(\delta)$ denote a function with a bounded upper symmetrical derived number $D^+\psi(x)$, then

$$f(\delta) = \int_\delta D^+\psi(x)$$

exists; and, since $f'(x) = D^+\psi(x)$ over δ^*, $f(\delta)$ and $\psi(\delta)$ differ* by a constant. In one dimension, this is Lebesgue's theorem that *a monotone increasing function with bounded derived numbers has a derivative almost everywhere.*

Theorem B can be extended by means of Vitali's covering theorem to any summable function, but we shall not stop to give here the details of the proof. The convergence of the series will be uniform over a set δ_1 differing from δ by as little as we please.

3. *Fundamental Theorems.* It will be readily seen that the following fundamental theorems follow at once from (1).

(1) The mean value theorem for integrals.

(2) The theorem that $\int_{\delta_1} f + \int_{\delta_2} f = \int_{\delta_1+\delta_2} f$, where δ_1 and δ_2 have only a nul-set in common.

(3) The theorem that $\int_\delta [f_1 + f_2] = \int_\delta f_1 + \int_\delta f_2$, proved by truncating f_1 and f_2, deriving, and then using Scheeffer's theorem.

(4) The theorem that $\lim\limits_{n\to\infty} \int_\delta f_n(x) = \int_\delta \lim\limits_{n\to\infty} f_n(x)$.

In conclusion we shall make some applications of Theorem A which will show its use in dealing with measurable functions.

4. *Application to Almost Everywhere Finite Measurable Functions.* If, in Theorem A, we cover each E_i set by a finite set of non-overlapping cells C_i such that

$$|\text{meas } E_i - \Sigma C_i| < \frac{\epsilon}{2^i},$$

and if we define a function equal to *one* over the C_i's and to *zero* over the complementary cells, we shall have a discontinuous function which can be made continuous by *trimming* the C_i cells or their complements and using suitable *connective tissue.* Thus we have the following theorem.

* In one dimension, this follows from Lebesgue's extension of Scheeffer's theorem, Lebesgue, *Leçons sur l'Intégration*, p. 79.

THEOREM C. *Any almost everywhere finite measurable function is equal to a continuous function save over a set of points of arbitrarily small measure* $\bar{\epsilon}$.*

If we denote this continuous function by $C(x)$, we have $\int_\delta F(x) = R\int_{\bar{\delta}} C(x) + \bar{\bar{\epsilon}}$ when δ and $\bar{\delta}$ differ by as little as we please, where $R\int_\delta$ denotes the Riemann integral.

5. *Egoroff's Theorem.* From § 4, we can deduce Egoroff's theorem:† *If a sequence of measurable functions* $F_i(x)$ *converges to a limit* $F(x)$ *over* δ *it will converge uniformly save over a subset of arbitrarily small measure.*

6. *Lusin's Extension of Weierstrass' Theorem.* In § 4, by using a set of $\bar{\epsilon}$'s converging to zero, we have the theorem: *Any in general finite measurable function is the limit almost everywhere of a sequence of continuous functions.* Hence we have also Lusin's‡ extension of Weierstrass' theorem, viz.: *Any measurable function is almost everywhere the limit of a sequence of polynomials.*

7. *Approximately Continuous Functions.* Making use of Denjoy's definition of *approximately continuous functions,*§ we have the theorem: *A measurable almost everywhere finite function is approximately continuous save for a nul-set.*

By means of Vitali's covering theorem it is easy to show that *any almost everywhere approximately continuous function is measurable.* Hence we can say that the R-integrable functions are almost everywhere continuous and the L-integrable functions are almost everywhere approximately continuous, when it is understood, of course, that the functions are limited.

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* Cf. Lusin, COMPTES RENDUS, June 17, 1912.

† COMPTES RENDUS, Jan. 30, 1911.

‡ Lusin, COMPTES RENDUS, loc. cit.

§ A function $F(x)$ is approximately continuous at x when it is continuous over a point set of metric density one at x. BULLETIN DE LA SOCIÉTÉ DE FRANCE, vol. 43, p. 165; or Hobson, vol. 1, p. 295.