THE MOMENT OF INERTIA IN THE PROBLEM OF N BODIES*

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The differential equations of the problem of n bodies are

$$m_i x_i'' = \frac{\partial V}{\partial x_i}, \; m_i y_i'' = \frac{\partial V}{\partial y_i}, \; m_i z_i'' = \frac{\partial V}{\partial z_i}, \; (i = 1, \cdots, n)$$

in which the potential function is

$$V = \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} k^2 \frac{m_i m_j}{r_{ij}}, \; (j \neq i).$$

The kinetic energy of the system is

$$T = \tfrac{1}{2}\sum_{i=1}^{n} m_i (x_i'^2 + y_i'^2 + z_i'^2).$$

The moment of inertia of the n bodies considered as point masses with respect to the origin is

$$I = \sum_{i=1}^{n} m_i (x_i^2 + y_i^2 + z_i^2).$$

On differentiating this expression for the moment of inertia twice with respect to the time, we find

$$\tfrac{1}{2} I'' = \sum_{i=1}^{n} m_i (x_i x_i'' + y_i y_i'' + z_i z_i'') + \sum_{i=1}^{n} m_i (x_i'^2 + y_i'^2 + z_i'^2).$$

The last term of this expression is twice the kinetic energy of the system, $2T$. The first term of the right member, in virtue of the differential equations of motion, may be written in the form

$$\sum_{i=1}^{n} \left(x_i \frac{\partial V}{\partial x_i} + y_i \frac{\partial V}{\partial y_i} + z_i \frac{\partial V}{\partial z_i} \right).$$

Since V is a homogeneous function of degree -1 of the quantities x_i, y_i, z_i, we have

$$\sum_{i=1}^{n} \left(x_i \frac{\partial V}{\partial x_i} + y_i \frac{\partial V}{\partial y_i} + z_i \frac{\partial V}{\partial z_i} \right) = -V.$$

Consequently the differential equation for the moment of

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inertia may be written in the form

$$\tfrac{1}{2}I'' = -V + 2T.$$

Since $T - V = \mathcal{E}$ is the total energy of the system, and is therefore constant, we can eliminate the kinetic energy from the preceding equation, and we obtain

$$\tfrac{1}{2}I'' = V + 2\mathcal{E}.$$

This equation was derived by Jacobi and discussed by him in considerable detail in his *Vorlesungen über Dynamik.*

Since the moment of inertia is homogeneous of degree $+2$ with respect to the quantities x_i, y_i, and z_i, and since V is homogeneous of degree -1, it follows that the product

$$VI^{1/2} = C$$

is a homogeneous function of degree zero, which in certain cases is an actual constant with respect to the time. In any event, C is a function which is always positive and which depends only upon the masses and their relative distributions, if we choose the center of mass as the origin, and not at all upon the size of the configuration. Eliminating V from the differential equation by the introduction of the function C, we have

$$I'' = 2CI^{-1/2} + 4\mathcal{E}.$$

As Jacobi pointed out, this equation shows that if $\mathcal{E}$ is a positive constant at least one of the bodies must recede to infinity, and therefore if the system is to be a permanent one, $\mathcal{E}$ must be negative. To exhibit this fact, we shall take $\mathcal{E} = -E$, and we shall suppose that we are dealing with a permanent system, so that E is positive. Then

$$I'' = 2CI^{-1/2} - 4E.$$

If C is a constant, this equation can be integrated. The first integral is

$$I'^{1/2} = \frac{2}{E}[C_1^2 - (C - 2EI^{1/2})^2],$$

the constant of integration being taken to be $2[C_1^2 - C^2]E^{-1}$. Evidently the new constant C_1^2 cannot be negative. If it is zero, $I^{1/2}$ has the constant value $I^{1/2} = C/(2E)$.

The second integral is

$$\frac{C}{(2E)^{3/2}} \sin^{-1}\left(\frac{2EI^{1/2} - C}{C_1}\right) - \frac{1}{(2E)^{3/2}} \sqrt{C_1^2 - (C - 2EI^{1/2})^2} = t - t_0,$$

which shows that $I^{1/2}$ is a periodic function of t with the period

$$P = \frac{2\pi C}{(2E)^{3/2}},$$

and that it oscillates about the mean value $C/(2E)$, the amplitude of the oscillation having the value C_1. It is worthy of note that the period of the oscillation is independent of the amplitude. It depends only upon the energy of the system and upon the nature of the configuration.*

In the problem of two bodies, the function C is necessarily a constant and has the value $(m_1m_2)^{3/2}/\sqrt{m_1 + m_2}$. In the problem of three bodies it is a constant for the equilateral triangular solution and for the straight line solution of Lagrange. It is constant for Longley's parallelogram solution of the problem of four bodies, and for Moulton's straight line solution of the problem of n bodies. In short it is constant for every solution of the problem of n bodies for which a definite geometric configuration is preserved throughout the motion, i.e., one in which the ratios of the mutual distances are constants.

This condition is sufficient, but there is nothing to indicate that it is necessary. In the globular star clusters we have a

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* If we replace $I^{1/2}$ in this integral by the radius of gyration, ρ, by means of the relation $I^{1/2} = \rho\sqrt{M}$ where $M = \Sigma m_i$; and make the further substitutions

$$\rho = \frac{C - C_1 \cos\varphi}{2E\sqrt{M}}, \qquad \frac{(2E)^{3/2}}{C}(t - t_0) = \tau - \pi/2, \qquad \frac{C_1}{C} = e,$$

it reduces to Kepler's Equation $\varphi - e \sin\varphi = \tau$, with

$$\rho = \frac{C}{2E\sqrt{M}}(1 - e\cos\varphi).$$

These equations are familiar in the problem of two bodies.

type of organization in which a steady state has apparently been reached. In such an aggregation, the function C must be very nearly constant over extraordinarily long intervals of time. If our galaxy is such a star cluster, it is oblate rather than globular. Assuming that it is composed of 1.5×10^9 stars such as our sun with an equatorial radius of 2000 parsecs (6,600 light years) and a polar radius of 600 parsecs, and that the mean velocity of the stars is 25 kilometers per second, the period of oscillation (if any exists) is 25,600,000 years and the period of a circular orbit about its equator is 74,300,000 years. If we assume a stellar density only one-fifth as great and a stellar velocity of 40 kilometers per second instead of 25 kilometers per second, the period of oscillation is 97,000,000 years, and the period of the circular orbit about the equator is 166,000,000 years. It should be added, however, that even though the function C is a constant (or nearly constant) for the galaxy, it is not necessary that there should be any oscillation. The moment of inertia may be a constant.

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SUBSTITUTIONS COMMUTATIVE WITH EVERY SUBSTITUTION OF AN INTRANSITIVE GROUP*

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It is well known how to find all the substitutions which are separately commutative with every substitution of a transitive group G of degree n and which involve no letter except such as are found in G. If G_1 is composed of all the substitutions of G which omit a given letter of G, and if the degree of G_1 is $n - \alpha$, then all the substitutions on the letters of G which are separately commutative with every substitution of G constitute a subgroup K of order α, and all the substitutions of K besides the identity are regular and of degree n.

When G is intransitive, the results are not quite so elegant, but as they are often useful it seems desirable to state them

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