SUMMABLE INFINITE DETERMINANTS*

BY WILLIAM L. HART

1. *Introduction.* It has been customary to define the value of the infinite determinant

(1) $$D = |a_{ij}|_{i,\, j=1,\, 2,\, \cdots}$$

by the equation

(2) $$D = \lim_{n\to\infty} D_n \qquad (D_n = |a_{ij}|_{i,\, j=1,\, 2,\, \cdots,\, n}),$$

in case the limit exists. Supposing that the a_{ij} are any complex quantities, we may state

DEFINITION. *The determinant* (1) *is summable to the value* D *in case the sequence* (D_n) *is summable to* D.

In this definition the term *summable* may be understood in any one of the various senses which have been given† to that term. It is easily seen that if D is summable, its value remains unchanged if we interchange its rows and columns. The sign of D is changed if two contiguous rows or columns are interchanged. If all the elements of a row, or column, are multiplied by a constant K, the new determinant obtained has the value KD.

It is the purpose of this note to present certain types of infinite determinants which are summable in the sense of the Cesàro first mean and, in the future, the unqualified word *summable* will be applied to determinants of this variety. In § 2, there will be given a proof of the summability and a discussion of certain properties of a class of determinants analogous to the Von Koch normal‡ infinite determinants. In § 3 the summability of a somewhat different type of determinants will be established.

2. *A Certain Class of Determinants.* A simple example of an infinite determinant which is summable to the value 1/2 is

* Presented to the Society at Chicago, March 26, 1921.

† E. Borel, *Les Séries divergents*, p. 87; Hurwitz, Report on divergent series, this BULLETIN, vol. 28, 1922, pp. 17–36.

‡ Cf. Kowalewski, *Determinantentheorie*, p. 372.

$$(3)\qquad D = \begin{vmatrix} 0 & 1 & & & & & \\ -1 & \pm 1 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & \pm 1 & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \cdot \end{vmatrix},$$

where all elements are zero except for those in the main diagonal minors of order 2, which are displayed, and where the $\pm$ signs in the main diagonal may be selected at will.

For (3), we find that $D_{2k} = 1$ and $D_{2k-1} = 0$ for $k = 1, 2, \cdots$. The determinant (3) would remain summable to the value 1/2 if arbitrary elements b_{ij} were substituted for the zero elements situated above the elements $+1$, in the columns of even index, and to the left of the elements -1, in the rows of even index.

Let the arbitrary quantities b_{ij} of the preceding paragraph be chosen as either $+1$, -1, or 0, and, after this selection, let the element of (3) in row i and column j be denoted by e_{ij}.

THEOREM I. *In* (1) *suppose that* $a_{ij} = e_{ij} + c_{ij}$, *where* $\Sigma_{ij=1}^{\infty} |c_{ij}|$ *converges. Then the determinant* D *of* (1) *is summable and*

$$(4)\quad 2D = D_2 + (D_4 - D_2) + \cdots + (D_{2h} - D_{2h-2}) + \cdots.$$

First let us consider certain properties of the infinite matrix

$$A = (c_{ij})_{ij=1,2,\cdots}.$$

By a minor of A of order p we shall mean a determinant of order p formed by those elements of a certain set of p rows of A which are located in a certain set of p columns. Let I represent the sum of the absolute values of all terms which enter in the expansions of all minors, of all orders, of the matrix A. Then it is well known that I converges.*

In order to establish the theorem, we note that the determinants D_n in the present case have the following properties when expanded in terms of the quantities c_{ij}:

(a) For every n, D_n is equal to the sum of certain terms from

* Cf. Kowalewski, loc. cit., p. 374.

I, with positive or negative signs attached, plus 1, if n is even, and plus zero, if n is an odd integer.

(b) For every value of $h > 1$, D_{2h} contains all terms of D_{2h-2}, and, moreover, the new terms occurring in D_{2h} do not occur in any D_{2k} if $k < h$.

(c) For every value of h, D_{2h-1} consists of terms which are not found, even with a change of sign, in any other determinant D_{2k-1} $(k \neq h)$.

The property (a) results from the fact that the elements e_{ij} are $\pm$ 1 or 0. One verifies (b) by expanding D_{2h} according to the elements

$$c_{1,\,2h-1},\, c_{2,\,2h-1},\, \cdots,\, c_{2h-1,\,2h-1},\, c_{2h,\,2h-1} - 1, \tag{5}$$

in its $(2h - 1)$-th column. We establish (c) by expanding D_{2h-1} according to the elements of its last column, which consists of all the quantities (5) except the last.

To establish the summability of D, we must show the existence of $\lim_{n\to\infty} S_n$ where

$$S_n = \frac{D_1 + \cdots + D_n}{n}. \tag{6}$$

When $n = 2k$ let us write

$$S_{2k} = \frac{R_{2k}}{2k} + \frac{T_{2k}}{2k},$$

where

$$R_{2k} = D_1 + D_3 + \cdots + D_{2k-1}$$

and

$$T_{2k} = D_2 + D_4 + \cdots + D_{2k}.$$

In view of properties (a) and (c) above it follows that $|R_{2k}| \leqq I$. Consequently,

$$\lim_{k\to\infty} \frac{R_{2k}}{2k} = 0.$$

It follows from properties (a) and (b) that each bracket in the series

$$K = D_2 + \cdots + (D_{2h} - D_{2h-2}) + \cdots,$$

when written in terms of the quantities c_{ij}, consists of terms from I with various signs attached. Since no term occurs in more than one bracket, it is seen that K converges. Therefore,

$$\lim_{k\to\infty} \frac{T_{2k}}{k} = K = \lim_{k\to\infty} D_{2k},$$

and it follows that $\lim_{k\to\infty} S_{2k} = K/2$. As a consequence of the simple identity

$$S_{2k+1} = \frac{R_{2k+2}}{2k+1} + \frac{T_{2k}}{2k}\frac{2k}{2k+1},$$

it is verified that $\lim_{n\to\infty} S_n = K/2$, which completes the proof of the theorem.

It is evident that the algebraic complement* A_{ij} of each element a_{ij} in the determinant of Theorem I is itself a determinant of the same type. For the complement is obtained by replacing a_{ij} by $+1$, or -1, as the case may be, and by substituting zero for all other elements in the ith row or in the jth column.

If we consider the determinant D' resulting from (3) by the addition of c_{ij} to the element in the ith row and jth column, for every i and j, it follows from Theorem I that D' is summable. Let E represent a determinant obtained from D' by replacing a finite number of its rows (or of its columns) by a bounded set of numbers. Then we could prove that E is summable by the method used in establishing Theorem I.

Let us consider the sequence $(t_n;\ n = 1, 2, \cdots)$ defined by

(7) $$t_{2k-1} = 2k, \qquad t_{2k} = 2k - 1 \quad (k = 1, 2, \cdots),$$

as the normal order for the arrangement of the positive integers. Then a sequence $p = (j_h;\ h = 1, 2, \cdots)$, containing all the positive integers, will be called an *alteration* of the order (7) if, for all values of h sufficiently large, $j_h = t_h$. Thus, for every alteration p, there exists a smallest even index $h = 2m$ such that $j_k = t_k$ if $k \geqq 2m - 1$. Consider the first $(2m - 2)$ terms of p,

(8) $$j_1, \cdots, j_{2m-2},$$

which give a permutation of the numbers $(1, 2, \cdots, 2m - 2)$. Let us call p an *even* or *odd* alteration according as (8) is obtainable from the order $(1, \cdots, 2m - 2)$ by an even or by

* Cf. Kowalewski, loc. cit., pp. 378, 382.

an odd number of successive interchanges of neighboring numbers. It is well known* that the evenness or oddness of p is independent of the way in which these interchanges were made.

To each alteration p let us make correspond the term

$$P_p = \pm \prod_{k=1}^{2m-2} a_{k,\ j_k} \Big[\prod_{h=m}^{\infty}(-\ a_{2h-1,2h}\, a_{2h,\ 2h-1})\Big],$$

where the $+$ sign is selected if p is even and the minus if p is odd. Suppose that (1) is a determinant of the type E mentioned above. Then it can be proved that

$$(9) \qquad 2E = \sum_{(all\, p)} P_p,$$

where the summation is extended over all alterations p of the system of positive integers. On the basis of this result, it can then be shown that, for every j and for every k,

$$(10) \qquad E = \sum_{i=1}^{\infty} a_{ij}A_{ij} = \sum_{j=1}^{\infty} a_{kj}A_{kj},$$

where A_{ij} is the algebraic complement of the element a_{ij} in (1). The proofs of (9) and (10) will not be given since the reasoning would be practically identical with that used in the derivation of similar results in the theory of normal infinite determinants.† By use of (9) and (10), it would be possible to develop a theory for the solution of the infinite system of equations

$$(11) \qquad \sum_{j=1}^{\infty} a_{ij}x_j = b_i, \qquad (i = 1, 2, \cdots),$$

provided that the determinant (1) formed by the coefficients a_{ij} is of the type D' defined above. A very casual inspection, however, shows that system (11) could easily be transformed into a system of a well known type whose solution could be obtained in terms of normal infinite determinants.

The determinants we have considered were obtained from (3) by simple transformations. If we should take more complicated determinants than (3) as our points of departure we could obtain results similar to those derived above. For example, consider the determinant of order q given by

* Cf. Kowalewski, loc. cit., p. 9.

† Cf. Kowalewski, loc. cit., §§ 154, 155.

$$(12) \qquad \begin{vmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & 0 & \cdots & \pm 1 & \\ & & & 1 & 0 & & \\ & & & & \ddots & \ddots & \\ & & & & & 1 & 0 \end{vmatrix} \begin{matrix} \\ \\ (p\text{th row}), \\ \\ \\ (q\text{th row}) \end{matrix}$$

where all elements not explicitly indicated are zero and where the ambiguous sign in the last column is selected in such a way that the determinant has the value $+1$. In the infinite determinant D of (1) let the elements a_{ij} in the rows of index $mq+1$, $mq+2$, $\cdots$, $mq+q$, for all values $m = 1, 2, \cdots$, be selected as zero, except for those in the columns $mq+1$, $mq+2$, $\cdots$, $mq+q$. Choose these q^2 elements as those of the determinant (12). It is easily verified that D is summable to the value p/q. With this determinant D as a starting point instead of the determinant (3), the same methods that were used above suffice to establish a theorem analogous to Theorem I.

The results stated after Theorem I would also have their analogies in the present case. Once more it should be noted that the determinants considered in this paragraph would not enable one to solve any problem in connection with infinite systems of linear equations which could not equally well be solved by the known theory of normal infinite determinants.

3. *Another Type.* An example of the type* of determinants to be considered in the present section is given by (1) in case

$$(13) \qquad a_{ij} = 0, \qquad (i \neq j); \qquad a_{ii} = (-1)^i, \quad (i = 1, 2, \cdots).$$

Then D is summable to the value zero and satisfies the equation

$$(14) \qquad D = \frac{1}{2}(\lim_{n\to\infty} D_{2n+1} + \lim_{n\to\infty} D_{2n}).$$

Another example of this type is given by (1) if

$$(15) \qquad \begin{gathered} a_{ij} = 0, \qquad (i \neq j); \\ a_{2i-1,\,2i-1} = \alpha, \qquad (a_{2i,\,2i} = \beta,\ i = 1, 2, \cdots;\ \alpha\beta = 1). \end{gathered}$$

In this case D satisfies (14), where the first limit is α and the second is 1, so that $D = (\alpha + 1)/2$. It is obvious that (15) may be generalized to the case where the main diagonal

* The author acknowledges his indebtedness to Professor L. L. Silverman for suggesting the consideration of the determinants treated in this section.

elements are $\alpha_1, \alpha_2, \cdots, \alpha_k, \alpha_1, \cdots, \alpha_k, \cdots$ and where all other elements are zero, under the supposition that

$$\alpha_1 \alpha_2 \cdots \alpha_k = 1. \tag{16}$$

If, under conditions (13), we should add c_{ij} to the element a_{ij} of D, for all (i, j), the resulting determinant would be summable and its value would be given by (14) in case the (c_{ij}) satisfied the condition of Theorem I. The proof of this statement would be almost a repetition of that of Theorem I.

Let D' represent the determinant resulting from (1), under conditions (15), by the addition of c_{ij} to a_{ij}.

THEOREM II. *In the sum I of Theorem* I *let p_k represent the sum of all terms containing exactly k factors c_{ij}. In* (15) *suppose that $|\alpha| > 1$ and that*

$$\sum_{k=1}^{\infty} p_k |\alpha|^k \tag{17}$$

converges. Then, D' is summable and satisfies the relation

$$D' = \frac{1}{2}(\lim_{n\to\infty} D_{2n+1}' + \lim_{n\to\infty} D_{2n}'). \tag{18}$$

The properties of summable sequences make it clear that the theorem will be completely established if we can show that the two limits in (18) exist. Let us consider the convergence of the sequence (D_{2n+1}'). The last two rows of D_{2n+1}' are:

$$\begin{matrix} c_{2n,\,1}\ , & c_{2n,\,2}\ , & \cdots, & c_{2n,\,2n} + \beta, & c_{2n,\,2n+1} \\ c_{2n+1,\,1}, & c_{2n+1,\,2}, & \cdots, & c_{2n+1,\,2n}, & c_{2n+1,\,2n+1} + \alpha. \end{matrix} \tag{19}$$

In expanding D_{2n+1}' by Laplace's Rule,* according to the minors of the last two rows, it is verified that we obtain

$$D_{2n+1}' = D_{2n-1}' + P_{2n+1}, \tag{20}$$

where P_{2n+1} consists of terms containing α, β and factors c_{ij}. Let the expression *frame of a term* of P_{2n+1} refer to the absolute value of the product of those factors c_{ij} entering in the term. The expression P_{2n+1} has the following properties:

(*a*) All terms contain at least one c_{ij} from those in (19) and hence the frames of terms of P_{2n+1} are distinct from those of P_{2h+1} for $h < n$.

(*b*) There are no repetitions among the frames of the terms of P_{2n+1}.

---

* Cf. Bôcher, *Introduction to Higher Algebra*, p. 24.

(c) After the relation $\alpha\beta = 1$ has been used, the power of α entering in a term of P_{2n+1} is at most one unit greater than the number of factors c_{ij} in that term. If a power of β remains after use of $\alpha\beta = 1$, the absolute value of the term is increased by neglecting the power of β because $|\beta| < 1$.

It is seen that the convergence of the sequence (D_{2n+1}') is equivalent to that of the series

$$(21) \qquad D_1' + P_3 + P_5 + \cdots + P_{2n+1} + \cdots.$$

Let $\bar{P}_{2n+1}$ represent P_{2n+1} with all of its terms replaced by their absolute values. As a consequence of the properties (a), (b) and (c) it follows that the convergence of (17) implies that of (21) because $\bar{P}_3 + \bar{P}_5 + \cdots \leqq |\alpha| \sum_{k=1}^{\infty} |\alpha|^k p_k$. Hence the sequence (D_{2n+1}') converges. A proof of the same nature as that just given would establish the convergence of the sequence (D_{2n}'). Therefore, Theorem II may be considered completely proved. If we had supposed $|\beta| > 1$, an analogous proof would have shown D' to be summable under the same assumption as was made in Theorem II.

Let E represent the determinant associated with (16) after c_{ij} has been added to the element in row i, column j, for all (i, j). The determinant E is summable if condition (17) is satisfied, where we understand $|\alpha|$ to represent the absolute value of the product of all factors α_h from (16) which satisfy $|\alpha_h| > 1$. The proof of this statement would be similar to that given for Theorem II. E would satisfy the equation

$$(22) \quad E = \frac{1}{k}\left(\lim_{n\to\infty} E_{kn} + \lim_{n\to\infty} E_{kn+1} + \cdots + \lim_{n\to\infty} E_{kn+k-1}\right).$$

Certain interesting questions regarding summable determinants remain unanswered in this note. It may be that Theorem II and its generalization, in connection with determinant E, are true under hypotheses less restrictive than those of this paper. More generally, it would be of interest to know whether a determinant (1), which is summable, remains so if quantities (c_{ij}) are added to its elements, where the c_{ij} satisfy the condition of Theorem I.

THE UNIVERSITY OF MINNESOTA