A PROPERTY OF CERTAIN FUNCTIONS
WHOSE STURMIAN DEVELOPMENTS
DO NOT TERMINATE*

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Let \( u_k(x) \) be a set of continuous Sturmian functions defined on the interval \((a, b)\), i.e., solutions of an equation of the form

\[
\frac{d}{dx} \left[ k(x) \frac{d}{dx} u(x) \right] + \lambda g(x) u(x) = 0,
\]

each satisfying two linear homogeneous self-adjoint boundary conditions and corresponding to a value of \( \lambda \) for which this is possible. We assume that the coefficients of the differential equation have derivatives of all orders, and that \( g(x) \) does not vanish in the closed interval \((a, b)\), nor \( k(x) \) on the open interval. Let \( f(x) \) denote a function with derivatives of all orders, satisfying boundary conditions to be specified presently. We proceed to call attention to a property which such functions must have if their developments in series of the functions \( u_k(x) \) are not to terminate.

We denote by \( a_k \) the \( k \)th generalized Fourier coefficient of \( f \):

\[
a_k = \int f u_k g \, dx,
\]

where we have omitted argument \( x \), and limits of integration, \( a \) and \( b \). No ambiguity will result from the abbreviation. If in the integral, we replace \( u_k g \) by its value obtained from the differential equation (1), and integrate by parts, we obtain

\[
a_k = -\frac{1}{\lambda_k} \int (k f')' u_k dx,
\]

where the integrated terms have been omitted on the assumption that \( f \) satisfies the same self-adjoint boundary conditions as the \( u_k \). Under this assumption, they vanish. We define a series of functions as follows:

\[
f_n = \frac{1}{g} \left( k f_{n-1}' \right)', \quad f_0 = f.
\]

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We now subject \( f \) to the hypothesis that it, and these derived functions, all satisfy the same boundary conditions as the \( u_k \).

It then appears that for all \( n \),

\[
a_k = \frac{1}{\lambda_k} \int f_n u_k g dx.
\]

This is a simple generalization of the long familiar equation for the coefficients of a Fourier series. It is rather in the inference drawn from it, than in the generalization itself, that the interest lies. From the equation (5) we pass to an inequality. Let \( B_k \) denote the maximum of \( |(b - a)g(x)u_k(x)| \), and \( F_n \) the maximum of \( |f_n(x)| \). Then, evidently,

\[
|a_k| \leq B_k F_n/|\lambda_k|^n, \quad \text{or} \quad F_n \geq (|a_k|/B_k)|\lambda_k|^n.
\]

It follows that unless the development of \( f(x) \) terminates, \( F_n \) must, for all \( n \) greater than a determinable number \( N \), exceed any exponential function of \( n, Ae^{pn} \), because of the known property of the characteristic numbers \( \lambda_k \) of having infinity as the only limit point of their absolute values.

In other words, if \( F_n \) is less than any such exponential function, for positive \( A \) and \( p \) and for infinitely many values of \( n \), \( f(x) \) is a homogeneous linear function of the \( u_k(x) \) with constant coefficients.

This property takes on special interest when \( h \) and \( g \) are constant, for in this case the \( f_n \) are proportional to derivatives of \( f \). With suitable boundary conditions, it then takes the form: if the periodic function \( f(x) \) has derivatives of all orders, it is either a trigonometric polynomial, or else the maximum of the absolute value of its \( n \)th derivative exceeds any exponential function \( Ae^{pn} \) for all \( n \) from a certain one on.*

In the case of analytic functions of a complex variable, we have the result:

Either \( f(z) \) is a polynomial, or else the maximum of the absolute value of its \( n \)th derivative on any circle lying entirely within its domain of analyticity exceeds any exponential function \( Ae^{pn} \) from a given \( n \) on.

\* The result in this particular form was announced to the Society Dec. 2, 1911. See this Bulletin, vol. 18, p. 234.