HAHN'S REELLE FUNKTIONEN

Theorie der reellen Funktionen. By Hans Hahn. Vol. I. Berlin, Julius Springer, 1921. vii + 600 pp.

A preliminary indication of the contents of a book may sometimes be conveyed by a statement as to what it does not contain. Such information is supplied by the preface to the present volume, which states that a second volume, completing the work, is to present the theory of integration and differentiation, the analytic representation of arbitrary functions, and Fourier's series. A lower inequality for the order of ideas involved is given by another statement in the preface, that although the principal facts of the general theory of sets and the theory of real numbers are summarized in an introduction for convenience of reference, no systematic development of these theories is attempted, and the difficult questions which gather around their foundations are not touched upon at all.

The prospective reader will be further enlightened by a glance at the author-index at the back of the book. There he will find thirty-eight references to Lebesgue, thirty-three to W. H. Young, and twenty-four each to Baire and Hausdorff, while, at the other end of the scale, Heine is mentioned four times, Cauchy three times, Dirichlet twice, and Riemann once.

The volume is concerned, then, with those investigations of the last few decades which have had the specific purpose of throwing the fullest and most searching light on the fundamental concepts of function and limit. In a larger sense, the working out of these concepts may be regarded as the supreme achievement of research in mathematical analysis during the past ninety or a hundred years. The various special theories, while offering more or less that is of value in their own particular conclusions, have derived an added significance from their relation to the central theme, the bringing of the mysteries of eighteenth-century and early nineteenth-century mathematics within the domain of assured knowledge and common-sense. Our files of journals doubtless contain thousands of pages which will receive scant attention from generations to come, but if this generalization of an existence theorem, or that simplification of a convergence proof, has contributed in its day to the stirring of ideas through which understanding emerges, it has done its part. From this point of view, the present work represents a somewhat comprehensive version, rather than a fragment, of what has been learned in a century of analysis. It is a summary too general in its terms, too abstract and refined, to be appreciated by limited human intelligence without a background of experience and illustration, which the reader must bring to its study; but it is an account possessing a certain symmetry and completeness of its own.

It goes without saying that a book which opens with the words "We begin with a brief survey of the simplest facts in the theory of sets," which presents the Wohlordnungssatz on page 25, and which in six hundred pages does not reach the subject of integration or differentiation, is not designed as an introductory text. An adequate appraisal of it as a work of science would require the insight of a critic much more widely and deeply read in the literature of the field than the present reviewer. His naïve impressions on turning its pages may, however, be of some value to other readers of similarly unspecialized qualifications.

As has been mentioned already, an introductory chapter summarizes the facts that are to be used subsequently with regard to sets in general, the transfinite cardinal and ordinal numbers, the real number system, and the various types of limits associated with sequences and sets of real numbers. There is no attempt to lay down a complete system of definitions and postulates. There is no definition of an infinite set, for example, and an irrational number is defined merely as a number that is not rational. The principle of choice is accepted with a simple statement that criticism of its logical basis need not be taken into account for the purpose in hand. It is used, or Zermelo's theorem based on it is used, without further question, when occasion demands, and at least once when it is not necessary, in the proof of the theorem that two sets are equivalent if each is equivalent to a part of the other. Nevertheless, one must not take too seriously the author's modest statement that the treatment is not "systematic." It will be an unusual reader whose knowledge of the subject is not materially improved both in range and in precision by a perusal of the chapter.

For the purposes of a study which is ultimately to be concerned largely with derivatives and derived numbers, it is convenient to permit functions to take on infinite values, and the real number system is formally extended in the present treatment by the adjunction of the two numbers $+\infty$ and $-\infty$. The careful working out of the idea shows clearly the difficulties with which any such definition has to contend. It is a matter of choice whether infinity shall be regarded as a number or not, but no definition can make it a number *like other numbers*. Nearly a page is devoted to an enumeration of the operations which may and those which may not be performed with the two infinite numbers. But complications are not thereby finally disposed of. For example, the operation $(+\infty) - (+\infty)$ is in general not defined (p. 28). But when the oscillation (Schwankung) of a function is defined, in a later chapter (pp. 190–191), it is stated that, for this particular purpose, $(+\infty) - (+\infty)$ shall be regarded as having the value 0. And yet, in a subsequent discussion of the same concept (pp. 214–215), it is found necessary to regard infinite values as exceptional again.

The limit of a sequence of numbers is so defined as to admit the possibility of infinite limits. "A sequence which has a limit is called convergent; in particular, if the limit is finite, the sequence is said to be properly convergent; a sequence that does not converge is called oscillating" (p. 32). The same convention is maintained in the definition of continuity (pp. 122–123), and even in those of uniform continuity (p. 131) and uniform convergence (pp. 246–247). On the other hand, when the limit of a sequence of points is introduced, while the definition that recognizes infinite limits is mentioned, it is not the one adopted for systematic use (pp. 56–57). The sequence of numbers 1, 2, 3, . . . , has the limit $+\infty$, while the corresponding sequence of points on the axis of reals (with the customary definition of distance) has no limit.

In calling attention to these contrasts, the reviewer does not wish to deprecate the introduction of infinite numbers. On the contrary, he regards the author's careful treatment of them, in a form for convenient reference, as a particularly valuable service to students of analysis. But it must be recognized that their use is at best the choice of the lesser of two considerable evils. The difficulties are in the nature of things, and are not lightly to be smoothed away.

Chapter I, on point sets, opens with the postulates for distance in a general metric space. The rest of the treatise is similarly general in scope, with occasional specific consideration of relations in euclidean space of one or more dimensions. This does not mean, however, that its aspect is uniformly forbidding to the general reader. On the contrary, it is singularly easy to dip in almost anywhere and gain at least a partial appreciation of what is going on. For example, one reads on page 86: "A part $\mathfrak{A}'$ of an arbitrary set $\mathfrak{A}$ is called a component of $\mathfrak{A}$, if $\mathfrak{A}'$ is connected, and every connected part of $\mathfrak{A}$ which has a point in common with $\mathfrak{A}'$ is a part of $\mathfrak{A}'$." The general intent of this is fairly obvious at a glance, although a precise understanding of it naturally involves a knowledge of the technical meaning of the terms involved; and if the reader turns back a few pages and finds that "A set $\mathfrak{A}$ is said to be connected, if it is not the sum of two non-vacuous parts, each closed on $\mathfrak{A}$," he may come to the conclusion that there is more in the definition first quoted than he thought. Of course it occasionally happens that one misses the point of a statement entirely, unless one has the exact meaning of the terminology in mind, as, in the latter definition, the distinction between *sum* (Summe) and *union* (Vereinigung) of two sets (p. 2). For the most part, however, be it repeated, a fair understanding of the text is surprisingly accessible even to the casual reader, such is the excellence of the presentation.

Chapter II deals with the concept of function, upper and lower bounds and limits of a function on a point set or at a point, continuity, and semi-continuity. It is to be remembered that the terms point and point set refer to elements and sets of elements in any kind of "space" which satisfies the necessary postulates. Mention has already been made of the admission of infinite values for a function; the definitions are of course framed for functions defined on an arbitrary point set, so that, for example, "at an isolated point of $\mathfrak{A}$ every function f is continuous on $\mathfrak{A}$" (p. 125), whether its value there is finite or infinite. Attention may be called to the sections on the continuous extension of a continuous function beyond its original domain of definition, and on correspondences between point sets, such correspondence being treated as a generalization of the function idea.

Chapter III is concerned with discontinuous functions, the oscillation functions associated with them, the distribution of points of discontinuity, and pointwise discontinuous functions in particular. The next chapter treats of sequences of functions: of continuous convergence at a point, uniform convergence at a point or over a point set, and related concepts. The fifth chapter gives a very systematic and satisfying account of the

Baire classification of functions. The last three chapters, on absolutely additive functions of sets, functions of limited variation, and measurable functions, cover a considerable part of what is commonly regarded as the theory of integration, though the process of integration itself is not discussed. The treatment of the measure of point sets in the sixth chapter is an especially interesting example of the method of postulates, several groups of theorems being developed in succession as successive restrictions are placed on the notion of measure.

The material appearance of the book is indicative of unremitting effort under adverse conditions. The paper, though fairly white and clear, is almost of the consistency of blotting-paper; the typography, on the other hand, is so excellent that a fairly attentive turning of the pages has disclosed only some half-dozen trifling misprints.

The exposition is remarkably clear and systematic throughout. If sometimes rather diffuse for a mere presentation of the facts, it is the more convenient for purposes of reference. The book is written neither as an exhibition of the author's learning, nor as a memorial to an abstractly conceived body of truth, orderly and symmetric as it is; it is written for the reader's information, by a man who anticipates the reader's difficulties and provides against them with great faithfulness and skill. If it be true that genius is nothing more than an infinite capacity for taking pains, this is unquestionably a work of genius; if it is possible to imagine a certain quality of inspiration which seldom thrills the reader of these pages, it is perhaps only because passages which would be regarded as evidence of inspiration in the case of a less eminently qualified writer are here accepted as a mere matter of course.

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