THE APRIL MEETING OF THE SOCIETY IN CHICAGO

The nineteenth western meeting of the Society being the fifty-first regular meeting of the Chicago Section, was held at the University of Chicago on Friday and Saturday, April 13 and 14, 1923. There were about eighty persons present at this meeting, among them the following sixty members of the Society:

Ballantine, E. M. Berry, Bliss, Blumberg, Brahana, Brink, C. C. Camp, Coble, Copenhaver, H. B. Curtis, Curtiss, Denton, Dickson, Dresden, Emch, Feldstein, Glover, L. M. Graves, W. L. Hart, M. G. Haseman, Hildebrandt, Hull, Ingraham, Dunham Jackson, Kazarinoff, Kinney, Krathwohl, Lefschetz, Logsdon, N. B. MacLean, MacMillan, Marshall, T. E. Mason, McEwen, J. V. McKelvey, G. A. Miller, Miser, E. H. Moore, E. J. Moulton, F. R. Moulton, C. I. Palmer, Pounder, Rechard, H. L. Rietz, Roman, Roth, Schottenfels, Sinclair, Slaught, Edwin R. Smith, Stetson, R. B. Stone, J. H. Taylor, E. L. Thompson, Townsend, J. S. Turner, Van Vleck, F. E. Wood, Yanney, J. W. A. Young.

The meeting was opened at 10 A.M. on Friday by Professor E. H. Moore, who presided during the first part of the morning session, after which Professor D. R. Curtiss took the chair. During the sessions of Friday afternoon and Saturday morning Professor A. B. Coble, Chairman of the Section, presided. On Friday afternoon Professor S. Lefschetz gave a symposium lecture on *Curves traced on algebraic surfaces*; this lecture will appear in the June number of this BULLETIN. Upon motion of Professor E. H. Moore, the Section expressed to Professor Lefschetz its appreciation of this lecture. The Section voted to hold the Christmas meeting of 1923 in Cincinnati, in affiliation with the meetings of the American Association for the Advancement of Science.

The papers read at this meeting are listed below. The paper of Mr. Davis was read by Professor Dresden. The papers of Blumberg and Barnett were read by title.

1. Professor H. L. Rietz: *On the representation of a certain fundamental law of probability of Laplace.*

This paper first develops the formula of Laplace for the frequency distribution of the sum of n elements each taken at random from an interval 0 to a by a simple geometrical method involving mathematical induction. It next shows that the approximation which Laplace used in his application of this formula to the problem of the random distribution of the inclination of the orbits of comets is given by the first two non-vanishing terms of the Gram-Charlier representation of an arbitrary continuous frequency function. The paper next develops a more accurate approximation from the Gram-Charlier representation in the sense of a certain least squares criterion. The main difficulty and much of the interest in the paper consists in obtaining remarkably simple expressions for moments of area under our theoretical frequency curve. The auxiliary theorems proved in this connection seem to be of interest in combinatorial analysis.

2. Professor F. E. Wood: *An application of areal coordinates to mixtures and the graphical solution of equations.*

In this paper the author derives the well known results involving the use of points within an equilateral triangle to represent mixtures of three substances; a different method of approach enables him to extend the representation to diluted mixtures of three substances and to general triangles. Some theorems, apparently new, are proved and a graphical method is obtained for finding the percentages of three given mixtures which need to be taken to form a desired mixture of the three fundamental substances. Another application is a method of finding graphically the solution of three simultaneous linear non-homogeneous equations in three unknowns.

3. Dr. H. R. Brahana: *A theorem concerning unit matrices with integral elements.*

Given a matrix M, the normal form of a skew-symmetric matrix of determinant 1 and with integral elements. Poincaré stated without proof that any matrix A with integral elements which satisfies the relation $AMA' = M$ can be considered as a product of matrices of two particular types. In this paper the theorem is proved and it is further shown that any matrix D which satisfies the relation $DND' = N$, where N is not the normal form, is equal to $B^{-1}AB$, B being unique for each N.

4. Dr. H. R. Brahana: *Riemann surfaces and the map problem.*

Heawood solved the map problem for a surface of genus 1, and pointed out the method of procedure for $p > 1$. There is required a "map of verification" for the minimum number of colors for each value of the genus greater than 1. Heffter gave these maps for each value up to 6. This paper translates the problem into one of selecting systems of branch points and cuts on a sphere so that the Riemann surface thus defined constitutes the required map on a surface of given genus. The solutions are given for genus equal to 1, 2, and 3. The paper also contains a definition of a Riemann surface based on analysis situs alone.

5. Mr. H. T. Davis: *An application of fractional differentiation to a class of Volterra integral equations.*

A theory of fractional differentiation is developed which removes a difficulty of Riemann's definition. The theory is then used to obtain solutions of Volterra integral equations of the second kind with infinite discontinuities in the kernel.

6. Professor G. A. Miller: *Same left co-set and right co-set multipliers for any given finite group.*

This paper will appear in full in an early issue of this BULLETIN.

7. Dr. M. M. Feldstein: *The invariants of the linear group modulo* p^k.

The n-ary linear homogeneous group modulo p^k, $G(n, p^k)$, is decomposed into factor-groups in two ways. The first decomposition brings to light the relations among the invariants of $G(n, p^k)$ and those of $G(n, p^{k-1})$. The second decomposition serves to derive the necessary and sufficient conditions which the invariants of $G(n, p^{k-1})$ must satisfy in order to be invariant under $G(n, p^k)$. Special treatment is required for $G(n, 2^k)$. A fundamental set of invariants is set down.

8. Professor S. Lefschetz: *Continuous transformations of manifolds.*

In this note there is given a new method for treating questions of continuous transformations of an n-dimensional manifold W_n. In particular it is shown how the determination of the minimum number of fixed points is reduced to a readily solved problem concerning intersections of cycles of a W_{2n}.

9. Mr. M. H. Ingraham: *Certain limitations of the value of the complete independence of a set of postulates.*

The notion of complete independence was introduced partly in order to do away with certain trick methods of securing ordinary independence. It eliminated the use of vacuous fulfilment of a postulate. Since then methods of questionable value, such as strengthening of postulates and introducing irrelevant elements have arisen for use in securing complete independence. Examples of such methods in recent papers are shown as well as examples against which there can be no criticism. The completely independent set of postulates for positive integers which was previously presented by the writer have such undesirable features and another set is presented from which these features have been eliminated.

10. Professor L. E. Dickson: *Foundations and status of the theory of linear algebras.*

In the usual definition of a linear algebra A of order n over a field F, each element of A is an ordered set $(x_1, \cdots, x_n)$ of n numbers of F. In recent work of Scorza (PALERMO RENDICONTI, vol. 45, p. 7, and *Corpi Numerici e Algebre,* Messina, 1921) the elements u_i of A are undefined entities, combined by addition and multiplication, and by a scalar multiplication producing products of u_i and numbers α_i of F, certain associative and distributive laws being postulated for these operations. Another postulate is that A contains n elements u_i such that every element of A is expressible in one and only one way in the form $\sum \alpha_i u_i$. The present paper replaces this strong postulate by the much weaker one that A has a finite basis. Various theorems are proved for algebras A over any field F which have been proved hitherto only when F is the field of all complex numbers. The results of this paper and those of the following paper will appear in the author's book entitled *Algebras and Their Arithmetics,* to be published in July by the University of Chicago Press.

11. Professor L. E. Dickson: *General theory of hypercomplex integers.*

Books on the arithmetic of quaternions were published by R. Lipschitz in 1886 and by A. Hurwitz in 1919. In a long series of extensive memoirs, Du Pasquier attempted to extend Hurwitz's methods to various algebras other than quaternions, but without success since under his definition most aglebras do not have integers, while if integers do exist, they are usually without interest since the laws of arithmetic fail and cannot be restored by the introduction of ideals however defined. This criticism is explained in full in the author's memoir in the JOURNAL DE MATHÉMATIQUES, 1923, where there is presented a new wholly satisfactory theory of integers in any algebra over the field of complex numbers. The present paper extends this theory to algebras over any field.

12. Dr. Irwin Roman: *The optical transformation of surface differentials.*

The problem of geometrical optics is that of studying the new wave train when we know the incident wave train and the refracting surface. The usual law of refraction is contained in the invariance of the vector product of $r_i h_i$ and H_i where r_i is the refractive index of the ith medium, h_i is the unit normal vector of the wave incident on the ith surface, and H_i is the unit normal vector for the interface between the ith and $(i + 1)$th media, at the point of incidence. The author combines a vector and a differential geometry method, obtaining the first and second differential coefficients of the refracting surface from the corresponding values for the interface and the incident wave, along with the distance function for this pair. For the special case of pure propagation, without refraction, the results furnish the usual formulas for parallel surfaces. The formulas simplify when the lines of curvature are taken as parametric curves, and also when the surface is one of revolution. A number of special cases are considered.

13. Professor F. R. Moulton: *Solutions of ordinary differential equations.*

If the right members of the differential equations are analytic in the dependent and independent variables, or if they have the Lipschitz property in the dependent variables and are continuous in the independent variable, then it is shown by a process depending on two parameters that the solution exists over the domain of definition of the differential equations. As a by-product the validity of the method of numerical solution of differential equations is established. It is shown by very direct means that the solutions and their first derivatives with respect to the independent variable are continuous functions of the independent variable and of the initial values of the variables, individually and conjointly. The extensions to the cases where the right members of the differential equations have continuous partial derivatives with respect to the dependent variables are included.

14. Professor Dunham Jackson: *On approximation by functions of given continuity.*

The problem of this paper, in its simplest form, is the approximation of an arbitrary continuous function $f(x)$ by means of functions satisfying a given Lipschitz condition. The discussion is concerned with the existence, uniqueness, and convergence properties of the function of closest approximation according to the criterion of least mth powers. The treatment is susceptible of extension in various directions.

15. Mr. D. Kazarinoff: *Vector treatment of the extrema of double integrals.*

In this paper it is shown that a double integral of the type studied by G. Kobb (Acta Mathematica, vol. 16 (1892)) with the integrand satisfying his conditions (4), (ibid., p. 69), can be reduced to $\int\int_S F(x, y, z, X, Y, Z)d\sigma$ and vice versa. Here X, Y, Z are the direction cosines of a normal to S and F is such that, k being any real constant, $XF_X + YF_Y + ZF_Z = kF$. This theorem is a consequence of one given by J. Radon, Monatshefte, vol. 22 (1911), pp. 53–63.

The author considers then the integral $\int\int_S F(r, n)d\sigma$ where F is a scalar function of r, a vector with components x, y, z, and n, the unit vector with components X, Y, Z, and $\text{grad}_n F\cdot n = kF$, and he derives a differential equation which a surface S must satisfy to furnish an extreme value for the integral. The author also considers some particular cases of this equation and its transformation into more explicit form, yielding a geometric interpretation.

16. Professor F. E. Wood: *Cubics associated with a net of conics.*

There is, in general, a unique cubic $K = 0$ associated with a net of conics, with the property that the first polars of the points of the plane with respect to $K = 0$ form the conics of the net. The equation of this cubic, the *fundamental cubic* of the net, and some of its properties, are derived.

If the point $P(\lambda_1, \lambda_2, \lambda_3)$ is associated with the conic $\lambda_1\varphi_1 + \lambda_2\varphi_2 + \lambda_3\varphi_3 = 0$ of a net of conics, then the points P which correspond to the degenerate conics of the net form a cubic curve in the λ-plane. This is called a Δ *cubic* of the net. In this paper the properties of the Δ cubics are obtained, and certain relations existing between the fundamental cubic, the Δ cubics and the Jacobian cubic of the net. The fundamental cubic of a net will, in general, uniquely determine and will be uniquely determined by a net of conics, which is true for the Jacobian cubic or the Δ cubics in special cases only.

17. Professor J. S. Turner: *An extension of the theory of the double modulus.*

In this paper, certain theorems of the classical double modulus theory are extended to the double modulus m, $P(v)$, where

m is any positive integer, and $P(v)$ is a rational and integral function of v, with integral coefficients, irreducible modulo m. Other theorems are extended to the double modulus p^2, $P(v)$, where p is a prime, and $P(v)$ is irreducible modulo p.

18. Professor Henry Blumberg: *On certain properties of sets of positive measure.*

Let $M_1, M_2, \cdots, M_n$ be n planar sets of positive measure (Lebesgue). It is then shown that there exist in the plane n circles $C_1, C_2, \cdots, C_n$ such that, if $P_1, P_2, \cdots, P_n$ are any n points lying respectively in these circles, there is a congruent set of points $Q_1, Q_2, \cdots, Q_n$ lying respectively in $M_1, M_2, \cdots, M_n$. Again, if M is a planar set of positive measure and $S = \{P_1, P_2, \cdots, P_n\}$, any finite set of points in the plane, there exists in M a set similar to S. Other properties are obtained. The results of the paper hold for n-space.

19. Professor I. A. Barnett: *On a class of invariant subgroups of the conformal and projective groups in function space.*

Kowalewski has defined the conformal group in function space as the totality of all regular infinitesimal transformations which leave invariant the angle between two curves in function space. In this paper all the subgroups of the conformal group are found which leave invariant the manifold in function space $\int_0^1 [f^n(x)/k(x)]\, dx = 1$. The explicit forms of these subgroups are given for the cases $n = 1, 2$. It is shown, furthermore, that for $n > 2$, there are no subgroups having the required property. An analogous discussion is made for the projective group of function space.

20. Professor I. A. Barnett: *The area-preserving group in function space.*

This paper studies all the regular infinitesimal transformations which leave invariant the areas of triangles in function space. These are $\delta f(x) = [\alpha(x) + \int_0^1 \beta(x, y)\, f(y) dy]\delta t$, where $\alpha(x)$ and $\beta(x, y)$ are arbitrary continuous functions of their arguments such that $\beta(x, y) + \beta(y, x) = 0$. These transformations form a group in the sense defined by Kowalewski which is, in fact, the analog of the group of motions in n-space. The preceding transformations also leave invariant all the curvatures of a skew curve in function space.

ARNOLD DRESDEN,
Secretary of the Section.