KLEIN'S COLLECTED PAPERS, VOLUME II.

Felix Klein: Gesammelte mathematische Abhandlungen. Vol. II: *Anschauliche Geometrie; Substitutionsgruppen und Gleichungstheorie; Zur mathematischen Physik.* Edited by R. Fricke and H. Vermeil. Berlin, Julius Springer, 1922. vi + 713 pages; 185 figures.

Although one of the editors has been replaced in the second volume, the purpose has not been altered, and it appears under the same plan as the preceding one. The memoirs in the first part, mentioned under the first subtitle, are sixteen in number, being memoirs XXXIV to XLIX inclusive. Of these, nine were written during the years 1873–6, five in the interval 1892–5, and two since 1900. This first part occupies 250 pages.

The first memoir is a description of four models of the Plücker complex surface, constructed while the author was a student and assistant of Plücker in Bonn. The first is the general Kummer surface with sixteen nodes, which appears as the surface of singularities of a general quadratic line complex. The second is the surface enveloped by lines of the complex which meet a given line. The line is double on the surface, which has eight double points. The third is defined in the same way, when the given line is an ordinary line of the complex, and the fourth has for double line a singular line of the complex. These models were rather crudely constructed and very crudely reproduced in metal casts, but their importance in the study of singularities of algebraic surfaces is very great. About two pages have been added to the original text, which was prepared for the *Katalog mathematischer Modelle* . . ., von W. Dyck (1892).

The next paper, of 52 pages, is on cubic surfaces. To the original paper that appeared in volume 6 of the MATHEMATISCHE ANNALEN, eighteen pages have been added, which explain more in detail the surfaces with biplanar points, and discuss the configuration of the 27 real lines on the diagonal surface. These two papers are the only ones on the construction of models, but they are sufficient to show the intimate relation between spatial intuition and mathematical conclusion as they existed in the mind of the author.

The procedure is nearly always the same, starting from a singular or indeed composite surface and seeing what changes necessarily follow when one or more parameters change slightly. The method was sharply attacked when first employed, then reluctantly tolerated on account of its unquestioned results, until now it is accepted as a completely rigorous and trustworthy process, brought to a wonderfully high degree of perfection by Italian geometers. The Rodenberg collection of models of cubic surfaces, proposed by Clebsch, was made more complete and systematic by the use of the above memoirs.

When Klein was called to Erlangen, in his opening address he proposed the establishment of a general collection of models and apparatus to simplify the teaching of mathematics. Later the same idea was developed much further in Munich, in collaboration with Professor A. Brill, whose brother L. Brill undertook the commercial manufacture of the models. While in Munich, Klein had for his assistant W. Dyck, who was an important factor in having models and apparatus recognized as an important and necessary part of mathematical instruction. The Brill factory in Darmstadt, later taken over by M. Schilling in Halle, under the scientific direction of Professor F. Schilling has supplied a large proportion of all the plaster models in use throughout the world. Since the death of Martin Schilling, and particularly since the war, the manufacture of a considerable number of the more complicated models has been discontinued. If this state of affairs is not corrected, it will mean a terrible loss to later generations of mathematical teachers.

The Dyck catalogue and a considerable collection of models was sent to Chicago as a part of the German exhibit at the World's Fair of 1893. Klein was the representative of the Prussian government, and was authorized to explain these models. Out of the talks and conferences occasioned by them grew the Evanston Colloquium.

The next two papers, on the connectivity of surfaces, and on relations among the singularities of algebraic curves, follow minutely the same ideas as those developed in the memoir on cubic surfaces. The most important element is spatial intuition. Now follow four papers, together over eighty pages, on the combination of analysis and spatial intuition, that are entirely convincing, are strikingly beautiful, and seem curiously simple when approached from this point of view. They are on a new form of Riemann surfaces, and their application to the study of Abelian integrals of genus 3.

The method is to study a composite quartic curve, consisting of two conics, and fix the rational parameters of the points of contact of the common tangents. Then replace the conics by a non singular quartic. Now bitangents appear, and points of inflexion, the arrangements of which can be partly determined immediately from the figure. The rational parameters are replaced by Abelian integrals, and the various cases are so clearly separated that one wonders why the subject can be so hard.

Much later (1892) the same ideas were applied to the normal curve of genus p. While the results here obtained are both striking and important, it soon becomes evident that the last word on the uniformization of algebraic curves has not yet been said.

A short paper on the discriminant variety of an algebraic curve, also prepared for the Dyck catalogue, and a note on the geometric interpretation of the successive convergents of a continued fraction complete the list of contributions to the study of space relations. The next forty pages are devoted to the theory of knowledge and to the foundations of mathematics. The point of view is that we learn by experience and perfect our knowledge by successive approximations. In his 1901–02 course on the applications of mathematics Klein advocated a theory of errors for graphical processes analogous to that of least squares for numerical data obtained from observation. Astronomers require extreme numerical nicety based on crude analytic formulas, yet the sequel has shown that they were most frequently right. The same phenomenon is observed in connection with the singularities of plane curves. Both the original formulas of Plücker and the further relations established by Klein were first established empirically and later justified by analysis without the use of figures.

The first paper concerns the concept of a function, the result being that, so far as graphical or numerical control is concerned, we must deal with a band rather than with a mathematical curve. The substance of this paper, presented in 1873, was further developed in the Evanston Colloquium, and still further in the later paper on the arithmetization of mathematics. In the attempt to have this point of view crystalized the Beneke prize was offered in 1898, but was not awarded, as no memoir of required merit was submitted. This was partly due to the vagueness in the statement of the prize problem. The desired direction is indicated in the next paper, but the most significant statement in it is the old slogan "there is no royal road in mathematics." The last paper is a short address on the border questions of mathematics and philosophy delivered in Vienna in 1906. It emphasizes the difference between intuition and demonstration.

The second part, finite groups of linear substitutions, occupies 250 pages, and includes twelve memoirs, and some dozen pages of comments. The first of these memoirs was written in 1871, and the last in 1905. The first incentive to study the problems here considered is found in the short note written by Klein and Lie on those curves (W-curves) having the property that every tangent meets the sides of the triangle of reference in points which with the point of contact have a constant cross ratio. The thought is that curves invariant under finite groups of linear transformations have special properties, the study of which will contribute both to our knowledge of the curves and of the groups to which they belong. The use of projective geometry and of the geometry of inversion is fully justified by the results these subjects produced in the *Ikosaeder* and later *Modulfunktionen*. In particular, the linear transformations of the complex plane and their interpretation in terms of rotations of the regular polyhedra form the key note of the earlier papers. The important papers of Schwarz, making use of the 120 alternately congruent and symmetric triangles on the surface of the sphere, defined by the icosahedron, and later the classic on the algebraic solutions of the hypergeometric differential equation, were eagerly studied and compared with his own interpretations. The study received a big impetus when Gordan came to Erlangen in 1874; for several years he was one of the most eager workers in this field. Although Klein left Erlangen for the technical school of Munich in 1875, intimate association with Gordan was continued until 1880, when Klein moved to Leipzig.

During these years Brioschi and Kronecker were occupied with the quintic equation. On account of the similarity of the form of results in the two inquiries, it was suggested to Klein that knowledge of the solvability of the quintic equation would be of use to him in the study of the icosahedron, but he proposed the converse problem, namely, to make the icosahedron the basis of the study of the quintic equation. By recalling the works of these men now it is fairly evident that each received great assistance from all the others, and there is no question that the geometric interpretation was an important factor for all of them. The most important illustration probably was the discovery of the simple ternary linear group of order 168, in connection with the transformation of order seven of the elliptic function, and the quartic curve which belongs to the group. How incomparably more instructive is the study of this curve than to confine one's attention to the general quartic. Klein refers to this period as the happiest and most fruitful of his life. But his intense activity began to undermine his health, and when he assumed new duties in Leipzig a break had to come. After a prolonged leave of absence, Klein realized that he would be obliged to work under lower pressure, and contented himself with systematizing and arranging results already obtained, rather than continuing the former program. In this period the *Ikosaeder* was written. Much of its substance had already appeared in memoirs, but in condensed form, and accompanied by considerable discussion of invariants. In the book the presentation is much more elementary, and most of the considerations of the theory of invariants is omitted. Klein protests against the idea that his book is only a visualization of a theory worked out by others; he maintains rather that it contains indispensable elements of the theory itself. At any rate Gordan succeeded in simplifying materially his own presentation after the book appeared. But Gordan was also able to simplify Klein's work on the sextic (MATHEMATISCHE ANNALEN, vol. 68 (1909)). This fact is graciously mentioned by Klein, who adds that the simplest and most elegant presentation is that given by Coble (MATHEMATISCHE ANNALEN, vol. 70 (1911)).

The study of the equation of the sixth degree was given to pupils, in particular to Reichardt * and to Cole.† Klein had made various attempts to employ quaternary linear groups in the study of the sextic equation, and dramatically describes the general surprise when Wiman called attention (MATHEMATISCHE ANNALEN, vol. 47 (1896)) to the existence of a simple linear ternary group of order 360, which had been found by Valentiner in 1889. Since this was first published in Danish, it was apparently entirely unknown until mentioned by Wiman, who showed that it was simply isomorphic with the even permutations of six letters. That the general solution of the sextic could be made to depend upon this group rather than upon a quaternary group was later proven by Klein in a letter to Castelnuovo. This letter was presented to the Accademia dei Lincei by Castelnuovo, and published in its RENDICONTI (ser. 5, vol. 8 (1899)). It appears in the present volume as memoir LX, and occupies page 480.

The last part of the volume, pages 505–713, is devoted to mathematical physics and contains memoirs LXII to LXXX inclusive. There are two

* *Ein Beitrag zur Theorie der Gleichungen sechsten Grades*, LEIPZIGER BERICHTE (1885), and *Ueber die Normierung der Borchardtschen Moduln der hyperelliptischen Funktionen vom Geschlecht* $p = 2$, MATHEMATISCHE ANNALEN, vol. 28 (1886).

† *A contribution to the theory of the general equation of the sixth degree*, AMERICAN JOURNAL, vol. 8 (1886).

sub-headings, one being Linear Differential Equations and the other General Mechanics. Klein had repeatedly stated that when he was a student his main interest was in physics, and that he deliberately planned an activity in mathematics for some years, as part of his preparation for work as a physicist. Although this plan could not be realized to any great extent, still, when we rehearse Klein's influence in furthering the knowledge and application of mathematical physics, we must concede that it was no mean accomplishment.

We are told in the introduction that the first stimulus was the association with Neumann while at Leipzig—1880 to 1886. The first papers were those on Lamé functions, closely followed by the monograph on Abelian integrals, all containing many physical concepts.

During these years Klein made several trips to France and to England, and he emphasizes how his ideas of mechanics were greatly clarified and extended by them, particularly through Maxwell and Hamilton. He takes pride in having had three important English books translated into German, Routh's *Dynamics*, Lamé's *Hydrodynamics* and Love's *Elasticity*. Two other books, written at a later time, were direct outgrowths of his courses, and have been standard books in their fields ever since. These are Pockel's *Ueber die Differentialgleichung* $\Delta u + k^2 u = 0$ *und deren Auftreten in der mathematischen Physik* (1891), and Bôcher's *Ueber die Reihenentwicklungen der Potentialtheorie* (1894). After Schwarz was called to Berlin (1892), Klein commenced a comprehensive plan of enlarging and broadening the mathematical instruction at Göttingen. Weber was called in 1892, succeeded by Hilbert in 1895, who largely directed the work in pure mathematics, while Klein devoted his energy to filling in the gap between mathematics and physics, assisted by Brendel, Sommerfeld, Riecke, Runge, and others. At this time Klein was sent to the Chicago exposition, and returned with American ideas of financing his new institute. Through the assistance of Althoff, minister of public instruction, he received liberal appropriations from the government, and through the assistance of Althoff as a private citizen he also gained the interest and generous financial support of several men prominent in the industrial world. By these two means he built up an extensive and symmetric institute, comprising a large number of branches of mathematics, and holding from its origin the front rank in the mathematical world. The encyclopedia was started in 1894; Klein suggested the plan, and personally assumed the editorship of the part on mechanics, which is just now nearing completion.

During the next decade he was active in so many organizations, and participated in so many international undertakings that his own productivity as an investigator decidedly diminished, yet as representative of the universities in the Prussian senate, as chairman of the International Commission on the Teaching of Mathematics, as member of the commission on the international catalogue of the Royal Society, and finally as chairman of the union of German teachers of science he exercised a tremendous influence in shaping mathematical instruction. His constant aim was not merely passively to understand the principles, but so to apply them as to make for continued progress, to intensify and enlarge life. Although these activities were all in the interests of applied mathematics, yet Klein has continued the directorship and editorship of the MATHEMATISCHE ANNALEN since the death of Clebsch; he completed fifty years in this office last November.

The memoirs on Lamé functions are followed by those on the zeros of the hypergeometric series, the representation of the hypergeometric function by means of definite integrals, and the auto-reviews of the autographed lectures on the hypergeometric function and the linear differential equation of the second order, given in 1893–4. These were all published in the MATHEMATISCHE ANNALEN. The remaining essays were all published elsewhere. They include: a short report on recent English investigations on mechanics, a discussion of space collineations which occur in optical instruments, the greeting given at the opening of the mathematical congress at Chicago, the Princeton sesquicentennial lectures, two papers on graphical statics, one on Painlevé's criticism of Coulomb's law of friction, and finally one on the formation of vortices in frictionless liquids. The list is followed by a detailed explanation of the causes which led to the respective studies. The third and final volume is now in press. It contains the memoirs in the theory of functions.

VIRGIL SNYDER

LÉVY ON FUNCTIONALS

Leçons d'Analyse Fonctionnelle. By Paul Lévy, avec une préface de J. Hadamard. Paris, Gauthier-Villars, 1922. vi + 442 pp.

The increasing importance which is being given to the theory of functionals, or functions of lines, is illustrated by the fact that three of the Borel monographs in the last ten years have been concerned with this branch of mathematics, and the great breadth of the subject is illustrated by the fact that there is so little overlapping between the most recent of these, which is the subject of this review, and the earlier ones by Volterra,* and the more recent Cambridge Colloquium Lectures by Evans. In his introductory chapter, Lévy makes an interesting distinction between "algèbre fonctionnelle" and "analyse fonctionnelle." The first includes problems in which the unknowns are ordinary functions, but where the methods of the theory of functionals are used in determining them. The second includes problems where the unknowns themselves are functions of lines, or where the problems themselves could not be considered independently of the notion of a functional. Most of the work of Volterra and Evans mentioned above would belong to the "algèbre." The present monograph is primarily concerned with the "analyse."

The idea of a continuous functional is of fundamental importance. A functional $U(x(t))$ is said to be continuous if $U(y_n(t))$ approaches $U(x(t))$

* *Leçons sur les Equations Intégrales et les Équations Intégro-differentielles,* reviewed by Westlund in this BULLETIN, vol. 20 (1914), pp. 259–62, and *Leçons sur les Fonctions des Lignes,* reviewed by Bliss, ibid., vol. 21 (1915), pp. 345–55.