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NOTE ON THE CONVERGENCE OF WEIGHTED TRIGONOMETRIC SERIES*

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1. Introduction. Let \( f(x) \) be a function continuous for all values of \( x \), and of period \( 2\pi \). Let \( T_n(x) \) be a trigonometric sum of the \( n \)th order.† If \( T_n(x) \) is determined, among all such sums, by the condition that the value of the integral

\[
\int_0^{2\pi} [f(x) - T_n(x)]^2 \, dx
\]

shall be a minimum, it becomes the partial sum of the Fourier series for \( f(x) \). The problem can be generalized by taking, as the quantity to be reduced to a minimum, the integral

\[
\int_0^{2\pi} \rho(x)[f(x) - T_n(x)]^2 \, dx,
\]

where \( \rho(x) \), indicating the weight to be attached to different values of the argument, is a function of \( x \), likewise of period \( 2\pi \), and positive for all values of \( x \). There is a considerable body of literature bearing more or less directly on the generalized problem. This literature owes its inspiration largely to the researches of Tchebychef;‡ particular mention should also be made of a classical memoir by Gram.§

The purpose of the following paragraphs is to discuss the convergence of \( T_n(x) \) toward the value \( f(x) \), as \( n \) becomes infinite. The method is one which I have used recently in connection with the corresponding problem in which the weight is constantly equal to unity, and the square of the error is replaced by a power with a different exponent. The

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† The words "of the \( n \)th order" will be understood throughout to mean "of the \( n \)th order at most."
‡ Cf., e.g., H. Burkhardt, Entwicklung nach oscillirenden Functionen und Integration der Differentialgleichungen der mathematischen Physik, Jahresbericht der Vereinigung, vol. 10, Heft 2 (1908), pp. 823 ff.
question of convergence is treated by Gram, in the paper cited, but scarcely in a manner to meet the requirements of modern analysis.* More recently it has come within the range of a number of investigations, including a series of papers by Stekloff, in the Bulletin de l'Académie des Sciences, Petrograd, and elsewhere, with which I am only very imperfectly acquainted; a paper by J. Chokhate,† which I have seen in manuscript; and a series of papers by Szegö.‡ Up to the present time, I have not seen any treatment covering precisely the results that are presented below. If it should appear nevertheless that such a treatment exists, the novelty of this paper would consist in the method employed, and in the applicability of the method to the case in which the exponent 2 in (1) is replaced by an arbitrary \( m \), as suggested in the concluding paragraph.

2. The Convergence Theorem. The conclusion to be established is as follows:§

Let \( \omega(\delta) \) be the maximum of \( |f(x') - f(x'')| \) for \( |x' - x''| \leq \delta \). Let \( \rho(x) \) be continuous and positive for all values of \( x \); or, if not continuous, let it be measurable, and always included between two fixed positive bounds. || Then we may state the theorem:

§ The proof of the existence of a unique solution for the minimum problem is based so directly on similar proofs already given that it will not be taken up in detail here; cf. D. Jackson, On functions of closest approximation, Transactions of this Society, vol. 22 (1921), pp. 117–128.
|| It would of course make no difference if this condition were violated at points of a set of measure zero, since the value of the integral (1), and consequently the determination of \( T_n(x) \), would not be affected.
The sum $T_n(x)$ will converge uniformly to the value $f(x)$ for $n \to \infty$ provided that

$$\lim_{\delta \to 0} \frac{\omega(\delta)}{\sqrt{\delta}} = 0.$$ 

As already stated, the proof is similar to one given recently in another connection.† In the first place, if $f(x)$ and $\varphi(x)$ are two functions whose difference is a trigonometric sum $t_n(x)$ of order $n$:

$$f(x) = \varphi(x) + t_n(x),$$

and if $T_n(x)$ and $\tau_n(x)$ are two sums, likewise of order $n$, such that

$$T_n(x) = \tau_n(x) + t_n(x),$$

the value of the integral (1) formed with $f(x)$ and $T_n(x)$ is the same as the value of the corresponding integral formed with $\varphi(x)$ and $\tau_n(x)$, and both integrals will reach their minimum values simultaneously. That is, if $T_n(x)$ and $\tau_n(x)$ represent the best approximating functions for $f(x)$ and $\varphi(x)$, respectively, as judged by the value of the integral (1), the errors $f(x) - T_n(x)$ and $\varphi(x) - \tau_n(x)$ will be identical.

By a general theorem on the approximate representation of continuous functions,‡ there will exist sums $t_n(x)$, of all orders $n > 0$, such that the difference between $f(x)$ and $t_n(x)$ never exceeds a constant multiple of $\omega(2\pi/n)$. In formulas, let

$$\varphi_n(x) = f(x) - t_n(x),$$

and let $\epsilon_n$ be the maximum of $|\varphi_n(x)|$; then

$$\epsilon_n \leq c\omega(2\pi/n),$$

where $c$ is independent of $n$. In particular, if $\omega(\delta)$ satisfies

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* There is no reason to suppose that the particular infinitesimal $\sqrt{\delta}$ has any essential significance for the problem; its occurrence is in all probability due merely to the limitations of the method.


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the hypothesis of the theorem,

\( \lim_{n \to \infty} \epsilon \sqrt{n} = 0. \)

Let \( \tau_n(x) \) be the trigonometric sum of order \( n \) which gives the best approximation to \( \varphi_n(x) \), as determined by the integral corresponding to (1); let

\[
\gamma_n = \int_0^{2\pi} \rho(x)[\varphi_n(x) - \tau_n(x)]^2 dx;
\]

and let \( \mu_n = |\tau_n(x_0)| \) be the maximum of \( |\tau_n(x)| \). Let it be assumed that

\[
0 < v \leq \rho(x) \leq V,
\]

the numbers \( v \) and \( V \) being constants; and let it be assumed temporarily that \( \mu_n \geq 4\epsilon_n \).

By Bernstein’s theorem,* since

\[
|\tau_n(x)| \leq \mu_n,
\]

it follows that

\[
|\tau_{n'}(x)| \leq n\mu_n
\]

for all values of \( x \). In particular, for values of \( x \) in the interval

\[
|x - x_0| \leq \frac{1}{2n},
\]

it can be inferred that

\[
|\tau_n(x) - \tau_n(x_0)| \leq \frac{\mu_n}{2},
\]

and

\[
|\tau_n(x)| \geq \frac{\mu_n}{2}.
\]

Since

\[
|\varphi_n(x)| \leq \epsilon_n \leq \frac{\mu_n}{4},
\]

it follows further that

\[
|\varphi_n(x) - \tau_n(x)| \geq \frac{\mu_n}{4}
\]

throughout the interval specified, and, as the length of the interval is \( 1/n \), and \( \rho(x) \geq v \),

\[
\gamma_n \geq \frac{v}{n} \left( \frac{\mu_n}{4} \right)^2.
\]

On the other hand, by the minimum property of $\tau_n(x)$, the value of $\gamma_n$ is less than that which would be obtained if $\tau_n(x)$ were replaced by any other trigonometric sum of order $n$; in particular, by comparison with the integral which is obtained if 0 is substituted for $\tau_n(x)$,

$$\gamma_n \leq 2\pi V e_n^2.$$ 

Hence

$$\frac{v}{n} \left( \frac{\mu_n}{4} \right)^2 \leq 2\pi V e_n^2,$$

$$\mu_n \leq 4 \sqrt{\frac{2\pi V}{v}} e_n \sqrt{n}.$$ 

This relation, derived on the hypothesis that $\mu_n \equiv 4e_n$, clearly holds in the contrary case also, since $V \geq v$ and $n \geq 1$.

In any case, then, since $|\varphi_n| \leq e_n$ and $|\tau_n| \leq \mu_n$,

$$|\varphi_n(x) - \tau_n(x)| \leq e_n + 4 \sqrt{\frac{2\pi V}{v}} e_n \sqrt{n} \leq k e_n \sqrt{n},$$

where $k$ is independent of $n$. But it has been pointed out already that $\varphi_n(x) - \tau_n(x)$ is the same as $f(x) - T_n(x)$, where $T_n(x)$ is the sum giving the best approximation to $f(x)$, as determined by the integral (1); hence

$$|f(x) - T_n(x)| \leq k e_n \sqrt{n}.$$ 

This relation, combined with (2), establishes the truth of the theorem.

With the same method of treatment, the problem can be varied by using a general power of the absolute value of the error, instead of the square, together with a weight-function $\rho(x)$; and the method is applicable also to problems of polynomial approximation. For treatment in detail, however, the case discussed above may be regarded as sufficiently illustrative.

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