MATHEMATICS FOR THE LAYMAN

Mathematical Philosophy. A Study of Fate and Freedom. By Cassius J. Keyser. New York, E. P. Dutton and Co., 1922. 14 + 466 pp.

An author who attempts to give a popular exposition of any technical doctrine, be it in science, in art, or in philosophy, who (to use Professor Keyser's own words) aims to make a "contribution to the democratization of science and scientific criticism," faces a peculiarly difficult dilemma. He must beguile the layman's interest without being superficial; he must achieve adequate depth without being dull. A work of popularization, moreover, to fulfill its highest mission must be not merely descriptive or informative, but also interpretative. No one would probably deny the crying need of our age for popularizing books in all fields of technical inquiry satisfying these demands. And yet how meager the supply!

Professor Keyser has succeeded in meeting these conflicting demands in a remarkable way. His lectures were prepared primarily for students of philosophy; his book is addressed, however, to "educated laymen" in general. The appeal of its interest is very wide. It aims to discuss "the nature of mathematics, its significance in thought, and its bearings on human life." A large program, carried through with notable success! This does not mean, of course, that the last word has been said, that there is not room for difference of opinion, that many important fields of inquiry have not been left untouched. It does mean, however, that Professor Keyser has made a very significant contribution to the solution of the problem he set himself and that he has set up a standard of excellence which books of similar purpose and scope will find it difficult to meet.

The introductory lecture contains a discussion of the fundamental aims of education which is one of the finest things in the book. Many of our influential educators would confer a boon upon their country were they to repair to a mountain top, far away from the tangled undergrowth of short-sighted policy, and there read and ponder this discussion and let their souls commune with the educational ideal there presented. Such an educator might return to the uncharted wilderness of practical problems with a new vision and a more reliable compass.

The next eight lectures deal with various aspects of postulate systems. The point of view is that of Russell and Whitehead and much emphasis is laid on the alleged identity of pure mathematics and symbolic logic. Personally I would question the desirability of arbitrarily restricting so well established a term as "pure mathematics" to the meaning implied in such identification. Why not call the class of doctrinal functions "abstract mathematics" or "formal mathematics," or, if one really believes in the alleged identity, "formal logic"? Quite aside from this verbal problem, however, it seems to me that there is a real distinction between the problem of formal logic and the investigation of the various doctrinal functions—

a distinction at least of motive which I think is vital and should not be obscured. The motive of the logician is logical consistency or the validity of mental processes as such, the motive of the mathematician is the conclusions reached by such processes, their validity being by him assumed. Professor Keyser himself is not altogether consistent in his treatment of this subject. We may well argue that pure mathematics and formal logic are "organically related as are the roots of a tree and its branches," without thereby asserting that the roots and the branches are "the same thing." Furthermore, the definition of mathematics as a class of doctrinal functions—especially in the older form of Russell as "The class of all propositions of the form P implies Q"—leaves out, it seems to me, an essential characteristic of mathematics, viz., its structure. Mathematics is not a mere class—it is at least an ordered class of some sort.

A first example to illustrate the possibility of difference of opinion. Some others may appear later in this review. Let me say immediately that these opportunities for questioning constitute one of the most valuable features of the book. It is stimulating and thought provoking to a high degree.

I have used the phrase "doctrinal function." It is a very happy one and constitutes, I venture to say, a permanent addition to the vocabulary of our science. In proposing it Professor Keyser has done more than merely give an appropriate label. The concept which it describes was doubtless latent in the minds of many before its introduction. But by giving it a name he has precipitated the concept in precise form—and the concept is a valuable one.

Throughout these first nine lectures he develops also the thesis implied by his subtitle "A Study of Fate and Freedom." He calls attention at the very beginning to the eternal quality of ideas and their interrelations, to the immutable laws of thought to which the intellect is subject. "The world of ideas is the empire of Fate" (p. 5). After having developed the concept of a postulate system he returns to his thesis as follows (p. 136): "When once the principles, or postulates, are chosen, the die is cast—all else follows with a necessity, a compulsion, an inevitability that are absolute —we are at once subject to a destiny of consequences which no man nor any hero nor Zeus nor Yahweh nor any god can halt, annul or circumvent. Mathematics is in a word the study of Fate. Let me hasten to say that the Fate is not physical, it is spiritual. . . . The Fate is logical Fate. Is it a tyrant? And the intellect, then, a slave? . . . Where then is the intellect's freedom? What do you love? Poetry? Painting? . . . Music? The muses are *their* fates. If you love them you are free. Logic is the muse of *thought*. When I violate it, I am erratic; if I hate it, I am licentious or dissolute; if I love it, I am free—the highest blessing the austerest muse can give."

Here and throughout the book the author makes out a strong case for Fate—but it seems to me he leaves the case for Freedom unnecessarily weak. So much could have been said—and said beautifully, eloquently, by this particular author. I am sorry he did not say it. Is it true that the die is cast when the postulates have been chosen? What of the wealth of definitions that are possible under a given set of postulates? What of the infinite variety of combinations of the primitive elements which the creative imagination can construct? Royce has called it the "eternal fairyland of mathematical construction." The author would perhaps challenge this point of view—would claim that the imagination does not *create* these combinations, that they are present from the start, and that the mind merely *discovers* them. Some of the author's remarks on the early pages of the book seem to imply this attitude. It seems to me, however, that such an attitude, while perhaps philosophically tenable, is logically unnecessary and esthetically unsatisfying. Did Beethoven discover his Fifth Symphony, or did he create it? Did Georg Cantor discover his Mengenlehre or did he create it? Was the group concept, were the manifold, beautiful developments in the recent theory of functions of real variables discovered or created? No dogmatic answer is perhaps possible —in that the answer depends on the philosophical temperament of the one who replies. But it seems to me at any rate far more satisfying, and equally true, to regard such conceptions as the creations of an artistic imagination. The recent work on the foundations seems to me to have contributed not a little to a better understanding of mathematics as a fine art, like music or painting. That Professor Keyser is fully sympathetic toward such a conception no one can doubt—it is, therefore, a bit puzzling that he does not make more of it in this book. The volume is in itself a fine example of the play of imagination, but it has little to say of imagination as such and the rôle it plays in mathematics.

After his treatment of postulate systems follow lectures on Transformation, Invariance, Group Concept, Limits, Infinity, Hyperspaces, Non-euclidean Geometries, Psychology, Korzybski's Concept of Man, and Science and Engineering. These lectures are so good that they deserve a critical and discriminating review. However, space forbids; and Professor Keyser has assured me that he would prefer a review of general character, as more in keeping with the purpose of the book. Furthermore, an adequate appreciation and criticism must be a composite from many sources. Let philosophers quarrel with the author, if they must, concerning his philosophical tenets; let scientists object, if they will, to the proposition that "mathematics is the prototype which every branch of science approximates in proportion as its basal assumptions and concepts become clearly defined" (but let them not venture into the argument until they have an adequate conception of the broad sense in which the term mathematics is used); let literary critics challenge, if they be so disposed, the author's analysis of the function and art of criticism; and let sociologists and engineers evaluate the author's conception of their proper function in accordance with Korzybski's idea of the nature of our human kind.

Enough has already been said to indicate my conviction that Professor Keyser has made a valuable contribution to the literature of popularization in the highest and best sense of this much abused term. His book accurately presents the more fundamental conceptions which form the basis of modern mathematics. Of this I am confident, even though I have read the book attentively (and pleasurably) rather than critically. I found little to criticize in the statement of fact—perhaps nothing that would not lay me open, in view of the book's purpose, to the charge of quibbling.* He has presented these conceptions with a wealth of illustration, and in a style that is always pleasing and often of rare beauty and power. He has developed many and often surprising connections and analogies with apparently remote fields of inquiry. I venture to say that no one, be he professional mathematician or educated layman, can read this book without feeling its stimulating and thought-provoking character, provided only he be philosophically minded. A man not interested in meditating on the general aspect of things would perhaps find the book dull. But what a lot of the joy of life such a man must miss.

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TWO TRANSLATIONS OF ARCHIMEDES

Les Œuvres Complètes d'Archimède. Traduites du Grec en Français avec une introduction et des notes. By Paul Ver Eecke. Paris and Brussels, Desclée, de Brouwer et Cie., 1921. lx + 553 pp.

Kugel und Zylinder von Archimedes. Uebersetzt und mit Anmerkungen versehen. By Arthur Czwalina-Allenstein. No. 202 of Ostwald's Klassiker der exakten Wissenschaften. Leipzig, Akademische Verlagsgesellschaft, 1922. 80 pp.

In considering these two recent evidences of Belgian and German scholarship it may naturally be asked why a new edition of the complete works of Archimedes, or even of a single treatise, should be thought worthy of publication at this time, particularly in view of the fact that we already have the monumental edition by Heiberg, with its recent revision; the

* In the interest of removing minor blemishes in a future edition, attention may be called to the following: In the group definition of the geometry of shape on p. 218 reference should be to "each and all the transformations of the similitude group" and *no others*; on p. 267, line 7 from the bottom, after the word "field" the restriction ($n \neq n'$) should be added; on p. 329 the statement that a plane of circles is "as rich in circles as in point-triads, as rich in circles as ordinary space in points" is erroneous unless the point-triads be restricted to those formed from points of a line, and is open to misunderstanding, since it leads rather easily to the erroneous idea that dimensionality is a function of the cardinal number of a class rather than of the arrangement of its elements. The extended treatment of the concept of limit seems to me unnecessarily involved and difficult; this portion of the book is hard reading even for one familiar with the concept.

Very few typographical errors were noticed. These occur on p. 136, line 5 from the bottom; on p. 175, line 4 from the bottom; on p. 243 lines 9 and 10; on p. 271, line 11; and on p. 377, line 12.