The Second Mean Value Theorem for Summable Functions

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We start with the following lemma: If \( f(x) \) is summable over the interval \((a, b)\) and hence has an indefinite integral \( F(x) \) over \((a, x)\), \( a \leq x \leq b \), then

\[
\lim_{h \to 0} \int_a^b \frac{f(x + h) - f(x)}{h} \, dx = \lim_{h \to 0} \frac{F(b + h) - F(b)}{h} - \lim_{h \to 0} \frac{F(a + h) - F(a)}{h} = f(b) - f(a),
\]

provided that \( f(x) \) is right (or left) continuous at the ends of the interval.

Now let \( \phi(x) \) denote a monotone function right (or left) continuous at \( a \) and \( b \), and consider the identity

\[
\int_a^b \frac{F(x + h)\phi(x + h) - F(x)\phi(x)}{h} \, dx = (I) + (II),
\]

where

\[
(I) = \int_a^b \frac{F(x + h) - F(x)}{h} \phi(x + h) \, dx,
\]

\[
(II) = \int_a^b F(x) \frac{\phi(x + h) - \phi(x)}{h} \, dx.
\]

Applying a well known theorem of Lebesgue's to (I), and the first mean value theorem to (II), since we know that the expression \([\phi(x + h) - \phi(x)]/h\) is always of the same sign when \( a \leq x \leq b \), we have

\[
F(b)\phi(b) - F(a)\phi(a) + \epsilon'' = \int_a^b f(x)\phi(x) \, dx
\]

\[
+ F(\xi_h) \int_a^b \frac{\phi(x + h) - \phi(x)}{h} \, dx + \epsilon'''
\]

*Presented to the Society, September 18, 1923.*
where $\varepsilon_h'$ and $\varepsilon_h''$ vanish with $h$. Applying the lemma to the last integral, we have:

$$F(b)\phi(b) - F(a)\phi(a) = \int_a^b f(x)\phi(x)dx + F(\xi_h)(\phi(b) - \phi(a)) + \varepsilon_h,$$

where $\lim_{h \to 0} \varepsilon_h = 0$.

This can be written in the form

$$\int_a^b f(x)\phi(x) = \phi(b) \int_a^\xi f(x) + \phi(a) \int_\xi^b f(x) + \varepsilon_h.$$

When $\varepsilon_h$ takes on the value zero, $F(\xi_h)$, which is continuous, will take on for some value $\xi$, lying between $a$ and $b$, a value such that

$$\int_a^\xi f(x)\phi(x) = \phi(b) \int_a^\xi f(x) + \phi(a) \int_\xi^b f(x),$$

which is Weierstrass’s form of the second mean value theorem for integrals.

If, for example, we suppose $\phi(x)$ increasing (monotonically) and replace $\phi(x)$ by $A$ (fixed) over the interval $(a, a + k)$ and by $B$ (fixed) over the interval $(b, b + k)$, leaving the values of $\phi(x)$ unchanged over $(a + k, b)$, we can prove in the same way that

$$\int_a^\xi f(x)\phi(x) = A \int_a^\xi f(x) + B \int_\xi^b f(x),$$

where

$$A \equiv \phi(x) \equiv B$$

over $(a, b)$, by applying the same reasoning and letting the parameter $k$ approach zero.

From this form the usual Bonnet forms of the theorem are at once deducible.

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