N and R. The initial form (1_2) becomes one or two new quadratic forms in N and R. We proceed similarly with a prime factor of a/p, etc. Finally, we obtain formulas for x from $ax = \xi - by$. We conclude that all integral solutions of (8) are products of the same arbitrary integer by the numbers obtained from a finite number of sets of four expressions each quadratic in four arbitrary parameters. The explicit formulas will be discussed on another occasion.

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ON THE REALITY OF THE ZEROS OF A λ-DETERMINANT *

BY R. G. D. RICHARDSON

Some of the best-known theorems of algebra are centered around the zeros of the polynomial in λ,

$$(1)\qquad \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} .$$

In the classical case of the determinant connected with the equations of secular variations, where the elements a_{ij} are real and the determinant $|a_{ij}|$ formed from (1) by omitting the λ's is symmetric $(a_{ij} = a_{ji})$, these zeros turn out to be real. This theorem concerning the reality of the zeros has been extended † to the case where a_{ij} and a_{ji} are conjugate complex $(a_{ij} = \bar{a}_{ji})$. It is proposed in this note to extend it to a still more general case which has arisen in some investigations concerning pairs of bilinear forms just completed by the author. This generalization consists in allowing the coefficients of the λ's to be n^2 in number instead of n as in (1), of allowing them to be various and complex instead of all unity, and of bordering the determinant by m rows and m columns. The

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† Cf. Kowalewski, *Einführung in die Determinantentheorie*, p. 130.

method of proof is extremely elegant and it is worthy of notice that for the special case of the known theorems concerning (1), the proof itself is much simpler than any previously given.

Let us consider the determinant*

$$
(2)\qquad \begin{vmatrix}
a_{11}-\lambda b_{11} & \cdots & a_{1n}-\lambda b_{1n} & a_{1,\,n+1} & \cdots & a_{1,\,n+m}\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
a_{n1}-\lambda b_{n1} & \cdots & a_{nn}-\lambda b_{nn} & a_{n,\,n+1} & \cdots & a_{n,\,n+m}\\
a_{n+1,\,1} & \cdots & a_{n+1,\,n} & 0 & \cdots & 0\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
a_{n+m,\,1} & \cdots & a_{n+m,\,n} & 0 & \cdots & 0
\end{vmatrix},
$$

under the following hypotheses:

(*a*) the elements a_{ij} are conjugate complex;

(*b*) the elements b_{ij} are conjugate complex and the corresponding hermitian form

$$\sum_{ij}^{1,n} b_{ij}z_i\bar{z}_j$$

is definite.†

Under these hypotheses it is proposed to show that the zeros of the $(n-m)$th degree polynomial (2) in λ are real.

PROOF. Consider the $n+m$ homogeneous equations

$$
(3)\qquad \begin{cases}
\displaystyle\sum_{j}^{1,n}(a_{ij}-\lambda b_{ij})x_j+\sum_{k}^{n+1,n+m}a_{ik}x_k=0 & (i=1,\cdots,n),\\
\displaystyle\sum_{k}^{1,n}a_{ik}x_k=0 & (i=n+1,\cdots,n+m),
\end{cases}
$$

in $n+m$ variables x, whose coefficients are the elements of the determinant (2). The necessary and sufficient condition that

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* It is obvious that m must be less than n; otherwise the polynomial would be identically zero, and any theorem concerning the roots necessarily trivial.

† A necessary and sufficient condition that it be positive definite is that the terms of the sequence of $n+1$ determinants

$$
1,\quad b_{11},\quad \begin{vmatrix} b_{11} & b_{12}\\ b_{21} & b_{22}\end{vmatrix},\quad \begin{vmatrix} b_{11} & \cdots & b_{13}\\ \cdot & \cdot & \cdot\\ b_{31} & \cdots & b_{33}\end{vmatrix},\quad \cdots,\quad \begin{vmatrix} b_{11} & \cdots & b_{1n}\\ \cdot & \cdot & \cdot\\ b_{n1} & \cdots & b_{nn}\end{vmatrix}
$$

all have the same sign; the condition that it be negative definite is that these terms alternate in sign.

(3) have a solution is that (2) vanish. Hence for any zero of (2), the equations (3) have a solution $x_1, x_2, \cdots, x_{n+m}$, not identically zero. Multiplying the ith equation (3) by $\bar{x}_i$ $(i = 1, \cdots, n + m)$, and adding, we find a relation which may be written in the form

$$\sum_{ij}^{1,n} a_{ij}x_j\bar{x}_i - \lambda \sum_{ij}^{1,n} b_{ij}x_j\bar{x}_i + \sum_{i}^{1,n} a_{ik} \sum_{k}^{n+1,n+m} x_k\bar{x}_i + \sum_{i}^{n+1,n+m} \sum_{k}^{1,n} a_{ik}x_k\bar{x}_i = 0. \tag{4}$$

Now, by hypothesis (a), the terms $a_{ik}x_k\bar{x}_i$, $a_{ki}\bar{x}_kx_i$ are conjugate and their sum is real. Hence the first and second sums in relation (4) are hermitian forms and necessarily real, each of the n^2 terms in each sum being matched by its conjugate in the same sum. Further, since

$$\sum_{i}^{n+1,n+m} \sum_{k}^{1,n} a_{ik}x_k\bar{x}_i \equiv \sum_{k}^{n+1,n+m} \sum_{i}^{1,n} a_{ki}\bar{x}_kx_i,$$

it follows that each of the nm terms in the third sum is matched with a conjugate among the nm terms in the fourth and hence that the sum of these two sums is real. Finally by hypothesis (b), the second sum is different from zero (being zero only when all the x's vanish). Hence λ is real.

THEOREM. *The zeros of the* λ-*determinant* (2) *are real provided hypotheses* (a) *and* (b) *are satisfied.*

In the special case of the determinant of secular variations, (3) and (4) respectively take the forms

$$\sum_{j}^{1,n} a_{ij}x_j = \lambda x_i, \quad (i = 1, \cdots, n); \qquad \sum_{ij}^{1,n} a_{ij}x_ix_j = \lambda \sum_{i}^{1,n} x_i^2.$$

The proof of the reality of the λ's is thus notably simple.

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