BIANCHI'S DIFFERENTIAL GEOMETRY


The first and second editions of this classic, whether in Italian or German, are so well known to all students of differential geometry, or should be, that an extended review of this edition would be an anachronism. When one turns over the pages he is reminded of the earlier editions, but when he compares it with the latter he observes much that is new in the development of the topics formerly treated. And so he decides that hereafter, when he wants to see what Bianchi says about a certain subject (and a student of differential geometry cannot afford not to do so), he will use the new edition.

The first ten chapters of the second editions, both Italian and German, are retained in this volume, but they have been developed to such an extent that they occupy half again as many pages.

The problem of reducing the linear element of a surface to the form
\[ ds^2 = du^2 + 2 \cos \omega \, du \, dv + dv^2 \]
is called the problem of Tschebychef by Bianchi, and the parametric curves the lines of Tschebychef. This problem was discussed briefly on page 401 of Volume 2 of the old edition, but the new treatment is more extensive. In fact, it is shown that such lines exist on any surface, the degree of arbitrariness being that of two arbitrary functions of one variable each (p. 159). More recently Bianchi† has proved that the tangents to either family of these lines, say \( v = \text{const.} \), at points of any curve of the other family are parallel in the sense of Levi-Civita with respect to the latter curve; moreover, this is a characteristic property of the lines of Tschebychef.

In Chapter V it is shown that the Christoffel symbol of the second kind formed with respect to the second fundamental form is the arithmetic mean of the corresponding symbols formed with respect to the linear element of the surface and of its spherical representation. Also there are certain results concerning spaces of constant Riemannian curvature of \( n \) dimensions (p. 259).

In 1869 Dini solved the problem of geodesic representation of two surfaces upon one another. In 1896 Levi-Civita extended the problem to spaces of any order. The results of Dini and their application to

* The first German edition was reviewed by J. K. Whittemore in this BULLETIN, vol. 7 (1901), pp. 431-442, and the second German edition by L. P. Eisenhart in vol. 18 (1912), pp. 411-418.

† Bollettino Unione Matematica Italiana, vol. I (1922), pp. 11-16.
the determination of pairs of quadrics in this relation have been added to Chapter VI.

When two surfaces are applicable, the curves on one corresponding to asymptotic lines on the other have been called *virtual asymptotic* by Bianchi. The differential equations determining such lines on any surface were derived by Darboux, who showed that the problems of finding these lines on a surface \( S \) and the surfaces applicable to \( S \) are equivalent. This theory is set forth at the close of Chapter VII; it appears at the same place in the second German edition but not in the Italian one.

In his beautiful theory of surfaces applicable to a quadric Bianchi made use of the theorem of Chieffi: If at the point of any asymptotic curve \( C \) upon a surface \( S \) applicable to a ruled surface \( R \) one draws tangents to the geodesics \( g \) of \( S \) which are deform of the generators of \( R \), the ruled surface \( R_1 \), formed by these tangents is applicable to \( S \) with \( g \) remaining rigid. Bianchi adds in § 153 an analytic proof of this theorem to the geometric proof given on page 4 of volume 3 and in § 126 of the second German edition.

Twelve years ago Sannia published several papers in the *Rendiconti di Palermo* dealing with the intrinsic definition of rectilinear congruences by means of two quadratic forms. Bianchi devotes §§ 187 and 188 to this theory and to an application to congruences whose mean evolute is a point.

The general theory of relativity has aroused a general interest in the differential geometry of Riemann spaces of any order. The last half of the first volume of the second Italian edition contains an excellent exposition of this theory. This is omitted from the volume under discussion with the exception of the chapter dealing with "pseudo-spherical geometry and its non-euclidean interpretation", which appears as Chapter XIV. However, the last half of the second volume will be devoted to general Riemann geometry, and we can expect that it will contain some of the recent additions to this live subject.

Chapter XI is devoted to surfaces with plane or spherical lines of curvature, which appeared formerly as Chapter XXI with some omissions and additions. Chapters XXII and XXIII of the second Italian edition, or XIV and XV of the German edition, dealing with minimal surfaces, appear with little change as Chapters XII and XIII.

Bianchi began his career as a student of pseudospherical surfaces, and the theory of these surfaces and their transformations owes to him its present state of perfection. This subject has always seemed particularly fascinating to him. And now here in the last two chapters of this volume we have his exposition of surfaces of constant curvature based upon the knowledge and experience of a lifetime. They are extremely well done.

Anyone who is familiar with the method of the moving trihedral,
introduced by Darboux, and who has appreciated its power in dealing with geometrical problems, must wonder why Bianchi has never made much use of it. He introduced the elements of it in his treatment of Weingarten’s method for the study of applicable surfaces (vol. II, p. 176), but seems not to have used it elsewhere. It appears as a note at the close of this volume. 

Bianchi is a past-master in the art of writing treatises, not only in the field most closely associated with his name, but in many other fields. And this book reveals him at his best. The reader has no grounds for criticizing the book as to clarity of statement, but the student who wishes to follow up a particular subject may regret the lack of references to many of the sources from which the cream has been taken.

L. P. Eisenhart

THE IMUK REPORTS AND LOREY ON INSTRUCTION IN GERMAN UNIVERSITIES


During the deliberations of the fourth International Congress of Mathematicians at Rome in 1908, steps were taken to organize an International Commission on the Teaching of Mathematics, the members of which were to prepare or procure reports on the methods and materials of mathematical instruction in different countries. Most of these reports were ready when the fifth International Mathematical Congress convened at the University of Cambridge in 1912, but several more have appeared since then. At this writing, 18 countries have published 294 reports containing over 13,500 pages. Germany has issued 53 reports with a total of 5571 pages; about one-fourth of this space is required by the United States for its 18 reports, and about one-sixth of the same space by each of the following countries: Austria for 13 reports; Great Britain for 39; Switzerland for 13; and Japan for 16 reports in two volumes. The reports of France cover nearly 700 pages. Of more modest dimensions are, in order of size, the reports from Belgium, Russia (including Finland),

* A new word IMUK has been coined and is used in German writings as an abbreviation for these last three words.