A CHARACTERIZATION OF SURFACES OF TRANSLATION*

BY E. P. LANE

Let the non-homogeneous cartesian coordinates x, y, z of an arbitrary point on a surface be given as analytic functions of two independent variables u, v by equations of the form

$$x = U_1 + V_1, \quad y = U_2 + V_2, \quad z = U_3 + V_3,$$

where U_1, U_2, U_3 are functions of u alone and V_1, V_2, V_3 are functions of v alone. Such a surface is called a surface of translation because it can be regarded in two ways as generated by the motion of a curve which is translated so that its various points describe congruent curves. It is the purpose of this note to give another characterization of surfaces of translation, based on some notions which have arisen in the study of projective differential geometry.

Since, at each surface point P, the tangents to the curves $u = \text{const.}$ and $v = \text{const.}$ separate the two asymptotic tangents harmonically,† it follows that the parametric net is a conjugate net. Therefore the tangents to the curves $u = \text{const.}$, constructed at the points of a fixed curve $v = \text{const.}$, form a developable surface. The point where the tangent to the curve $u = \text{const.}$ through P touches the edge of regression of this developable is one of the two *ray points* of P, with respect to the parametric conjugate net. The other ray point is similarly defined on the other parametric tangent. The line joining these two ray points is called the *ray* of P. The totality of rays of all the points on the surface constitutes a congruence called the *ray congruence* of the fundamental conjugate net. And the

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† Lie, MATHEMATISCHE ANNALEN, vol. 14 (1879), pp. 332–337.

curves on the surface which correspond to the developables of the ray congruence are the *ray curves*. These definitions serve for any conjugate net on an arbitrary surface, and are due to Wilczynski.

But since our surface is a surface of translation and the parametric conjugate net is the net of generating curves, it follows that the developable circumscribing the surface along a curve $u = \text{const.}$, or a curve $v = \text{const.}$, is a cylinder. Therefore the ray points of every surface point are both at infinity, and the ray of every surface point is entirely at infinity. The ray congruence consists therefore of lines all lying in one plane, namely, the plane at infinity, and the ray curves are indeterminate.

One of my students, Mr. M. L. MacQueen, in his master's thesis, University of Wisconsin, 1923, has studied the class of surfaces on each of which there exists a conjugate net with indeterminate ray curves. He shows that the rays of such a conjugate net all lie in one plane, which we may call the *ray plane* of the net. He shows, furthermore, that every surface on which there exists a conjugate net with indeterminate ray curves can be projected into a surface of translation by precisely the projective transformation which carries the ray plane into the plane at infinity.

We may therefore characterize surfaces of translation by saying that *surfaces of translation are the only surfaces that have the property that on each of them there exists a conjugate net with indeterminate ray curves and with ray plane at infinity.*

THE UNIVERSITY OF CHICAGO