CONCERNING A SUGGESTED AND DISCARDED GENERALIZATION OF THE WEIERSTRASS FACTORIZATION THEOREM*

BY L. L. DINES

1. *Introduction.* A basic theorem in the theory of analytic implicit functions, proven by Weierstrass, may for our purpose be stated as follows:

Let (1°), $f(y; x_1, \ldots, x_n)$ *be analytic at the origin and vanish there*; *and* (2°), $f(y; 0, \ldots, 0)$ *be not identically zero. Then, throughout a certain neighborhood of the origin, there holds an identity of the form*†

$$(1) \qquad f(y; x_1, \ldots, x_n) = (P_0 y^m + P_1 y^{m-1} + \cdots + P_m)\, g(y; x_1, \ldots, x_n)$$

where $g(y; x_1, \ldots, x_n)$ *is analytic and does not vanish at the origin*; *and where* $P_j, j = 0, 1, \ldots, m,$ *is an analytic function of* $x_1, x_2, \ldots, x_n$ *and for* $j > 0$ *vanishes when* $x_1 = x_2 = \cdots = x_n = 0$.

In his *Madison Colloquium Lectures,* Osgood called attention to the fact that the hypothesis (2°) may be omitted in the case of a function $f(y; x)$ of only two variables, without disturbing the validity of the conclusion; and suggested tentatively but without proof that the theorem in this stronger form might be true for a function $f(y; x_1 \ldots, x_n)$ of $n+1$ variables. In a later paper‡ he showed very definitely that the theorem is *not* true in general, with the omission of the hypothesis (2°). His proof of this fact consisted in the exhibition of a function of the form

$$(2) \qquad f(y; x_1, x_2) \equiv x_1 - x_2 F(y)$$

which is not factorable in the form (1).

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† The theorem of Weierstrass states, in addition, that $P_0 = 1$, and that m is equal to the degree of the term of lowest degree in the series $f(y; 0, \ldots, 0)$.

‡ TRANSACTIONS OF THIS SOCIETY, vol. 17, page 4.

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The object of the present note is to consider further the possibility of an identity of form (1) in case (2°) is not satisfied. In the next article, we shall derive a simple necessary condition (Theorem I) for the existence of such an identity, and we shall show in particular (Corollary II) that a function of the form (2) is not factorable in the manner under consideration unless $F(y)$ is a *rational* function of y.

2. *A Necessary Condition.* If an identity of form (1) holds, then there is determined a second identity

$$\Phi \cdot f = P \tag{3}$$

where $\Phi = 1/g$, and $P = P_0 y^m + P_1 y^{m-1} + \cdots + P_m$. Let us suppose the power series expansions of the functions Φ, f, and P to be written in the following forms:

$$\begin{aligned} \Phi(y; x_1, \ldots, x_n) &= \Phi^{(0)} + \Phi^{(1)} + \Phi^{(2)} + \cdots \\ f(y; x_1, \ldots, x_n) &= f^{(k)} + f^{(k+1)} + f^{(k+2)} + \cdots \\ P(y; x_1, \ldots, x_n) &= P^{(k)} + P^{(k+1)} + P^{(k+2)} + \cdots \end{aligned} \tag{4}$$

where $\Phi^{(i)}, f^{(i)}$, and $P^{(i)}$ are homogeneous polynomials of degree i in $x_1, \ldots, x_n$. The polynomials $\Phi^{(i)}$ and $f^{(i)}$ have coefficients which are power series in y, while the coefficients of $P^{(i)}$ contain y to no higher power than the m th. Since g is analytic and does not vanish at the origin, it follows that $\Phi^{(0)}$ has a constant term different from zero. From the identity (3) it follows that f and P must begin with terms of the same degree in $x_1, \ldots, x_n$, and since we are considering the case in which (2) is not satisfied, i. e., in which $f(y; 0, \ldots, 0) \equiv 0$, we shall assume that this degree k is greater than zero.

Substituting the series (4) in (3), and equating polynomials of lowest degree on the two sides we obtain the identity

$$\Phi^{(0)} f^{(k)} = P^{(k)}. \tag{5}$$

A homogeneous polynomial of degree k contains power products of the form $x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$ where $j_1 + j_2 + \cdots + j_n = k$. Let the sequence of all such power products in any definite order be denoted by $X_1, X_2, \ldots, X_N$. Then we have

$$\begin{aligned} f^{(k)} &= f_1^{(k)} X_1 + f_2^{(k)} X_2 + \cdots + f_N^{(k)} X_N, \\ P^{(k)} &= P_1^{(k)} X_1 + P_2^{(k)} X_2 + \cdots + P_N^{(k)} X_N, \end{aligned}$$

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where $f_i^{(k)}$ is, for every i, a power series in y, while $P_i^{(k)}$ is, for every i, a polynomial in y. Substituting these expressions for $f^{(k)}$ and $P^{(k)}$ in (5) and equating coefficients of like power products, we find that

$$\Phi^{(0)} f_i^{(k)} = P_i^{(k)}, \quad i = 1, 2, \ldots, N.$$

Therefore

$$\Phi^{(0)} = \frac{P_1^{(k)}}{f_1^{(k)}} = \frac{P_2^{(k)}}{f_2^{(k)}} = \cdots = \frac{P_n^{(k)}}{f_N^{(k)}}; \tag{6}$$

or otherwise expressed, for every i and j for which $f_i^{(k)}$ and $f_j^{(k)}$ are different from zero,

$$\frac{f_i^{(k)}}{f_j^{(k)}} = \frac{P_i^{(k)}}{P_j^{(k)}} = \textit{a rational function of } y.$$

Hence we may state the following theorem.

THEOREM I. *If the leading homogeneous polynomial in* $x_1, x_2, \ldots, x_n$ *of* $f(y; x_1, \ldots, x_n)$ *be*

$$f^{(k)} = f_1^{(k)} X_1 + f_2^{(k)} X_2 + \cdots + f_N^{(k)} X_N$$

where $X_1, X_2, \ldots, X_N$ *are the various power products of degree* k *in* $x_1, x_2, \ldots, x_n$ *and* $f_1^{(k)}, f_2^{(k)}, \ldots, f_N^{(k)}$ *are power series in* y, *then a necessary condition that* f *admit factorization in the form* (1) *is that the ratio formed from each pair of non-vanishing coefficients* $f_i^{(k)}, f_j^{(k)}$ *be a rational function of* y.

COROLLARY I. *If any one of the sequence of coefficients* $f_1^{(k)}, f_2^{(k)}, \ldots, f_N^{(k)}$ *is a rational function, then all of these coefficients must be rational if* $f(y; x_1, \ldots, x_n)$ *is to admit factorization in form* (1).

COROLLARY II. *A necessary condition that the function*

$$f(y; x_1, x_2) \equiv x_1 - x_2 F(y)$$

admit factorization in the form (1) *is that* $F(y)$ *be a rational function of* y.

3. *Sufficient Conditions.* If the function $f(y; x_1, \ldots, x_n)$ is homogeneous in $x_1, \ldots, x_n$, that is if $f = f^{(k)}$, the condition of our theorem is *sufficient*. For if there exist poly-

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nomials in y, viz. $P_1^{(k)}, P_2^{(k)}, \ldots, P_N^{(k)}$ such that $f_1^{(k)} : f_2^{(k)} : \cdots : f_N^{(k)} = P_1^{(k)} : P_2^{(k)} : \cdots : P_N^{(k)}$, there will exist such polynomials in which the lowest powers of y are respectively equal to the lowest powers of y in the series $f_1^{(k)}, f_2^{(k)}, \ldots, f_N^{(k)}$. Hence there will exist a series $\Phi^{(0)}(y)$ with non-vanishing constant term satisfying (6). It is then easy to construct the polynomial $P^{(k)}$ satisfying the relation $\Phi^{(0)} f^{(k)} = P^{(k)}$. In view of the homogeneity of f this relation may be written in the form $\Phi f = P$. From this may be obtained the equivalent form (1).

If $f(y; x_1, \ldots, x_n)$ is not homogeneous in $x_1, x_2, \ldots, x_n$ the condition of the theorem is *not sufficient.*

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THE CLASS NUMBER RELATIONS IMPLICIT IN THE DISQUISITIONES ARITHMETICAE*

BY E. T. BELL

1. *Introduction.* The point of this note is its moral, which is to the effect that in arithmetic attention to trifles sometimes leads to beautiful and recondite truths. In particular, certain important expansions of Kronecker and Hermite relating to the number $F(n)$ of uneven classes of binary quadratic forms of negative determinant $-n$ are implicit in § 292 of the *Disquisitiones Arithmeticae* of Gauss, and might have been read off from there at a glance by anyone familiar with the *Fundamenta Nova* of Jacobi, thirty years before Kronecker first came upon them by the devious route of complex multiplication. The relevant trifle in this instance is changing the sign of an arbitrary constant throughout an algebraic identity.

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