FOUR BOOKS ON SPACE

Der Raum: Ein Beitrag zur Wissenschaftslehre. By Dr. Rudolf Carnap. Berlin, Reuter und Reichard, 1922. 87 pp.

Mathematik und Physik: Eine erkenntnistheoretische Untersuchung. By E. Study. Braunschweig, Friedrich Vieweg und Sohn, 1923. 31 pp.

Die realistische Weltansicht und die Lehre vom Raume: Zweite Auflage; Erster Teil. By E. Study. Braunschweig, Friedrich Vieweg und Sohn, 1923. x + 83 pp.

Mathematische Analyse des Raumproblems. Vorlesungen gehalten in Barcelona und Madrid. By Dr. Hermann Weyl. Berlin, Julius Springer, 1923. vii + 117 pp.

Kant described our knowledge of space and time as synthetic and a priori. By *synthetic,* he distinguished it from the analytic, more or less tautological judgments of abstract logic, while by *a priori,* he signified that it is independent of the concrete content of our senses. His account of the extensional properties of the universe was an attempt to bridge the gap between the purely abstract, ratiocinative science of geometry, and the obviously empirical nature of the space-world to which it applies.

While Kant's problem still exists, the last century has seen a tremendous change both in our notions of geometry and in our notions of the spatial world. Geometry no longer means Euclid, for since the days of Bolyai and Lobachevski we have become aware that there are other possible systems which yield no whit to the traditional geometry in the matter of logical rigor. The axioms of geometry signify no longer self-evident indubitable truths, but arbitrarily set assumptions. In short, from the mathematical standpoint, geometry is but a branch of logic, and like the rest of logic, is concerned with the consistency, the deductive sequence of its theorems, not with their truth. On the other hand, the universe is no longer treated as fitting primarily into the euclidean scheme, but into the more complicated schemes of the special or the general Einstein space-time system. For all this, the problem still remains as to how we can associate with our empirically known world of sense a mathematical structure which in at least its analysis situs properties is essentially that of Euclid, and in particular, how we can perform this association in a preliminary fashion, not merely as the final result of a long chain of careful experiments, but automatically, almost at first glance. This problem, as to the nature of our knowledge of space and time, has indeed become far more acute because of the demands made by the Einstein theory on our imagination. It has led to a large and important literature, of which the books and pamphlets here reviewed form a significant part.

That of Carnap is written most distinctly from the standpoint of the professional philosopher, but shows a far deeper acquaintance than common with the mathematical and mathematico-logical advances of recent years. It is clearly written, and contains a good description of the various types of spaces and geometries recognized by the mathematicians, together with a valuable bibliography. Carnap tries to retain the spirit of the Kantian treatment of space, while discarding those details which the progress of the last century has shown to be indefensible. He distinguishes three levels of spatial knowledge, and three types of space to which they respectively pertain. The formal space of the mathematician is on the same epistemological basis as pure logic. It is known a priori, and there is nothing in its theorems that has not been put in by free assumption in its postulates. On the other hand, the space of the physicist is susceptible to an experimental investigation differing in no fundamental way from that by which we study the phenomena of heat or sound or electricity. Our knowledge of this space is inductive and a posteriori. Mediating between these two spaces is the realm where Carnap hopes to preserve what is of permanent value in the Kantian philosophy. This is the so called space of intuition, which is known as a condition of physical experience, independently of the amount of physical experience which we possess. The developments of the last century have made it impossible for us to attribute to metrical or even to projective geometry the importance of intrinsic properties of intuitional space, but (so Carnap maintains) the analysis situs properties of the universe are known as Kant considered all space to be known: a priori, but synthetically, and with more than purely logical content.

It is hard to believe that the analysis situs properties of the universe, even *im Kleinen,* are forever to be immune to a criticism of the type which has led to the theory of relativity. Relativity itself, indeed, questions our conventional notions of the *im Grossen* connectivity of the world. There are signs that the time may not be far distant when the atomicity which the quantum theory recognizes as a basal characteristic of the universe shall be referred to a fundamentally atomic conception of its space-time framework. Veblen indeed has put forward the suggestion that physics may come to describe the world by a modular space. The analysis situs of such a space—if it may properly be said to have any analysis situs—is immeasurably different from that with which we are familiar.

Like Carnap, Study attacks the problem of mediating between the deductive science of mathematics, which includes such parts of mathematical physics as concern themselves with the formal development of the conclusions of assumptions not called into question in the course of the investigation, and the inductive science of experimental physics. Except that he is concerned with mathematics and experimental physics more broadly than as to their spatial aspects, Study's notion of the objects and methods of these disciplines does not depart widely from that of Carnap. The characteristic tool of the intermediate discipline, however, he conceives to be idealization. This idealization is of the nature of a *fiction*, a schematization, a diagram, which replaces the unmanageably complex properties of the empirical world by a working model of similar but simpler nature, satisfying the postulates of some known mathematical system. It is not of purely logical character, but involves a judgement of value, for similarity *in et per se* is an empty notion. It consists, as Study says, in dissecting out a single component of a complicated situation, and analyzing it apart from disturbing factors.

The theory of idealization certainly comes very close to the facts of our knowledge of space and of the external world. A not dissimilar view is held by Whitehead and by Russell. These authors have tried to develop the machinery by which we schematize the space and time of our experience—the space of bodies and the time of events—into the point-space and instant-time of theoretical physics. Study is unable to accept their treatment of this matter because of a fundamental epistemological difference.

Study is a realist. That is, he maintains that things have a reality entirely apart from their being perceived. Now, Russell is a realist also, but the realities of Russell are of the nature of sense-data, at least in large measure, and while they exist independently of being known, and may conceivably be shared by several observers, they are primarily given as objects of the experience of a single observer. In order to build out of these data the external world of physics, Russell makes the working hypothesis of the existence of other observers. Study points out that this hypothesis is of a precisely similar character to that of the existence of the external world of physics *tout simple.* Study accordingly starts his realism with this hypothesis. Like all hypotheses, he holds that it is capable of being confirmed or refuted by its consequences, whether it can be tested directly or not. It is clearly to be distinguished from a fiction, which is an assumption made for the purpose of conceptually simplifying a complicated situation, and with a full consciousness of its falsity.

Study, however, is fully appreciative of the value of Russell's analytical work. With all its incompleteness, its logical props and supports, it does represent a definite contribution to our understanding of what really is contained in our hypotheses concerning the universe. One can only regret that Study has made no similar attempt to dissect out the precise content of the hypothesis of the existence of the external world. One thing at least is certain: that the hypothesis of the external world, taken as a simple and indivisible supposition, has no explanatory force whatever. It is only the hypothesis of a very special sort of external world which fills the need which Study finds. The nature of this external world needs elucidation and analysis of the Russellian type.

Study's *Die realistische Weltansicht und die Lehre vom Raume,* in addition to a clear and forceful exposition of his realism, contains much of a polemic nature. He combats the Neokantians, the Pragmatists, the Conventionalists such as Poincaré, and many schools besides. All of his discussion is valuable, for like Poincaré and unlike the generality of philosophers, Study has the firm basis of a real acquaintance *in concreto* with the universe which he analyses *in abstracto,* combined with a generous measure of dialectic skill. There are, however, places where one feels that the vehemence of his wrath might have been tempered with a little more regard for the amenities of scientific discussion. Nothing is gained by speaking in a contemptuous tone of the "*merkwürdige Psyche*" of William James.

Weyl's book brings us back again to the geometry of the mathematician. While he recognizes in his introduction the trichotomy of geometrical space, the world of bodies, and some third intermediate space of physics, he makes no attempt to elucidate the nature of this transitional realm. His purpose is purely to develop the structural characteristics of the space of mathematics, more especially in its differential aspects. While he avoids an explicit employment of the postulational method, his work is postulational in spirit. It consists in showing the conditions imposed on the metrical nature of space by certain suppositions of a very abstract and general character. His fundamental notion is that of *affiner Zusammenhang.* He conceives of space as consisting at every point of a sheaf of infinitesimal vectors. Given a system of coordinates, it is possible to relate the sheafs at neighboring points by means of the "parallel translation" of Levi-Civita. A method which enables us to select from all the possible parallel translations of P_1 to the neighboring point P_2 a particular "genuine" one, whatever P_1 and P_2 may be, determines what he calls the *affiner Zusammenhang* of space. In the case of the geometry of Riemann, where each linear element has once for all a length which is a quadratic form in its components, the *affiner Zusammenhang* of space is determined by the condition that genuine parallel translations preserve length *im Kleinen.* In Weyl's generalized differential geometry, where the lengths of two vectors at different points can only be compared by translating one to the other along some path, and in which the result of comparison depends in general on the path, the *affiner Zusammenhang* is still completely determined by the system of measurement.

Weyl takes up at some length his geometry, in which the lengths of the different vectors at each point are still given relatively by the square root of a quadratic form in their components, but the relative change in the length of a vector under a small displacement of its origin, its components remaining the same, is given as a linear form in the components of displacement of the origin.

Weyl exhibits differential expressions, respectively known as the *Streckenwirbel* and the *Vektorwirbel,* the vanishing of which gives the condition that space be Riemannian or euclidean, as the case may be. He also shows that the familiar non-euclidean geometries of Lobachevski and of Riemann are the only homogeneous metrical geometries: the only geometries, that is, in which there is no intrinsic metrical characteristic by which any two points may be distinguished. To this homogeneity he assigns a certain epistemological significance; namely, that if any particular metrical structure of space be given *a priori,* it must be a homogeneous one.

Up to this point Weyl has been recognizing the quadratic form as fundamental for the determination of a system of measurement. The question of course arises, "Why not a quartic or sextic form?" In the case of a homogeneous space, following Helmholtz, Weyl shows that the condition that free mobility about a point be possible limits our differential form of distance to the square root of a quadratic form. In the more general case of an arbitrary metrical space, he here proves for the first time the exceedingly important theorem that the only possible measure of the length of all infinitesimal vectors which determines invariantly the volume of an infinitesimal parallelopiped with a point P as vertex and which permits the unique determination of an *affiner Zusammenhang* by the condition that distances are preserved, is the square root of a non-degenerate quadratic form in the components of the vectors.

It will be seen that, like the postulationalists, Weyl is interested in penetrating to certain very general propositions which lie at the basis of geometry. He is fortunate, however, in not being bound in the pedantic straight-jacket of independence-proofs and postulate-counting which has strangled a most promising young science almost in its cradle. Weyl is not behind the postulationists in rigor, but he is far ahead of them in imagination, and he relegates his meticulous dissection of logical minutiae to the place where it belongs,—the back of the book.

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