THE JACOBIAN OF A CONTACT TRANSFORMATION*

BY E. F. ALLEN

The equations

(1) $x_1 = X(x, z, p), \quad z_1 = Z(x, z, p), \quad p_1 = P(x, z, p),$

where X, Z, and P are functions of class C'', represent a transformation of line-elements in the xz plane to line-elements in the x_1z_1 plane. With Lie we shall define every transformation in x, z, p, which leaves the Pfaff differential equation

$$dz - pdx = 0 \tag{2}$$

invariant, as a contact transformation of the xz plane to the x_1z_1 plane. Hence the equations (1) must satisfy an identity of the form

$$dz_1 - p_1dx_1 = \varrho(dz - pdx), \tag{3}$$

where ϱ is a function of x, z, and p alone.

The following relations connecting X, Z, P, and their partial derivatives are easily obtained:†

$$\begin{cases} Z_x - PX_x = -p\varrho, \\ Z_z - PX_z = \varrho, \\ Z_p - PX_p = 0; \end{cases} \tag{4}$$

$$\begin{cases} [XZ] = X_p(Z_x + pZ_z) - Z_p(X_x + pX_z) = 0, \\ [PX] = \varrho, \text{ and } [PZ] = \varrho P. \end{cases} \tag{5}$$

The jacobian of transformation (1) is

$$J = \begin{vmatrix} X_x & X_z & X_p \\ Z_x & Z_z & Z_p \\ P_x & P_z & P_p \end{vmatrix}. \tag{6}$$

---

* Presented to the Society, December 1, 1923.

† Lie und Scheffers, *Geometrie der Berührungstransformationen*, p. 68, Chap. 3.

We shall show that this jacobian is equal to ϱ^2. Let us multiply the first row by P and subtract the product from the second row; then

$$(7)\qquad J=\begin{vmatrix} X_x & X_z & X_p \\ Z_x-PX_x & Z_z-PX_z & Z_p-PX_p \\ P_x & P_z & P_p \end{vmatrix}.$$

Hence, using equations (4), we find

$$(8)\qquad J=\begin{vmatrix} X_x & X_z & X_p \\ -p\varrho & \varrho & 0 \\ P_x & P_z & P_p \end{vmatrix}.$$

This reduces to

$$(9)\qquad J=\begin{vmatrix} X_x+pX_z & X_z & X_p \\ 0 & \varrho & 0 \\ P_x+pP_z & P_z & P_p \end{vmatrix},$$

when the second column is multiplied by p and the sum is added to the first column. Evaluating this determinant, we have

$$(10)\qquad J=\varrho[P_p(X_x+pX_z)-X_p(P_x+pP_z)].$$

Therefore, by equation (5), we may write

$$(11)\qquad J=\varrho^2.$$

The equations of a contact transformation may be regarded as the equations of a point transformation, which transforms points in xzp space to points in $x_1z_1p_1$ space. In general a surface in xzp space, represented by the equation $F_1(x, z, p)=0$, will be transformed into a surface in $x_1z_1p_1$ space, represented by the equation $F_2(x_1, z_1, p_1)=0$. Or if we regard equations (1) as the equations of a transformation of line-elements, it will transform a differential equation in x, z, p, into one in x_1, z_1, p_1, and also the solutions of the first differential equation into the solutions of the second.

Now if we set ϱ equal to zero,* we will have the equa-

* In some cases there are no values of the variables that will make ρ equal to zero. The following theory does not apply to such cases.

tion of a surface in xzp space, or we might say that we have a differential equation in xz space. Let us see into what this surface or into what this differential equation is transformed when it is subjected to the transformation (1). A few examples result in obtaining curves in $x_1z_1p_1$ space or in obtaining differential equations free from p_1. This leads to the following theorem.

THEOREM. *The surface* $\varrho = 0$ *is transformed into a curve in space by the transformation* (1).

If the partial derivative of ϱ with respect to z is not identically equal to zero, the equation $\varrho = 0$ may be solved for z.* Assuming that this is true, when the value thus obtained for z is substituted in X, Z, and P, they become functions of x and p alone. Regarding x and p as parameters, the equations (1) are the parametric equations of a surface. A necessary and sufficient condition† that

$$(12) \qquad x_1 = f(x, y), \quad y_1 = g(x, y), \quad z_1 = h(x, y)$$

define a curve in space and not a surface is that

$$(13) \qquad EG - F^2 = A^2 + B^2 + C^2 \equiv 0,$$

where

$$(14) \quad A = \frac{\partial(y_1, z_1)}{\partial(x, y)}, \quad B = \frac{\partial(z_1, x_1)}{\partial(x, y)}, \quad C = \frac{\partial(x_1, y_1)}{\partial(x, y)}.$$

To prove our theorem it is necessary and sufficient to show that the A, B, and C connected with equations (1) are identically equal to zero. That is, it is sufficient to show that all the determinants of the following matrix vanish identically:

$$(15) \qquad \begin{vmatrix} X_x + pX_z & X_p + \dfrac{\partial z}{\partial p} X_z \\ Z_x + pZ_z & Z_p + \dfrac{\partial z}{\partial p} Z_z \\ P_x + pP_z & P_p + \dfrac{\partial z}{\partial p} P_z \end{vmatrix}.$$

* If $\partial\rho/\partial z \equiv 0$ we will be able to solve for either x or p if $\rho \neq$ const.

† Eisenhart, *Differential Geometry*, p. 71.

Let us see what the effect will be when the value of z as obtained from $\varrho = 0$ is substituted in equations (1). Suppose that the substitution has been made in X and Z. It is easy to see that X_z and Z_z are equal to zero, and that to differentiate X completely with respect to x, it is necessary to differentiate with respect to x and then to use the function of a function rule, thus $X_x + X_z(\partial z/\partial x)$, and similarly for the other letters. Thus using the fact that $\varrho = 0$, we may write the equations (5) in the form

$$(16) \begin{cases} \left(X_p + \frac{\partial z}{\partial p} X_z\right)(Z_x + pZ_z) - \left(Z_p + \frac{\partial z}{\partial p} Z_z\right)(X_x + pX_z) = 0, \\ \left(P_p + \frac{\partial z}{\partial p} P_z\right)(X_x + pX_z) - \left(X_p + \frac{\partial z}{\partial p} X_z\right)(P_x + pP_z) = 0, \\ \left(P_p + \frac{\partial z}{\partial p} P_z\right)(Z_x + pZ_z) - \left(Z_p + \frac{\partial z}{\partial p} Z_z\right)(P_x + pP_z) = 0. \end{cases}$$

It is very easy to see that these equations are now the expanded form of the determinants of the matrix (15). Hence the theorem is proved.

THE UNIVERSITY OF MISSOURI

INTEGRO-DIFFERENTIAL INVARIANTS OF ONE-PARAMETER GROUPS OF FREDHOLM TRANSFORMATIONS*

BY A. D. MICHAL

1. *Statement of the Problem.* The author† has already considered functionals of the form $f[y(\tau_0'), y'(\tau_0')]$ (depending only on a function $y(\tau)$ and its derivative $y'(\tau)$ between 0 and 1) which are invariant under an arbitrary Volterra one-parameter group of continuous transformations. The

* Presented to the Society, December 1, 1923.

† Cf. *Integro-differential expressions invariant under Volterra's group of transformations* in a forthcoming issue of the ANNALS OF MATHEMATICS. This paper will be referred to as "*I. D. I. V.*"