Let us see what the effect will be when the value of $z$ as obtained from $q = 0$ is substituted in equations (1). Suppose that the substitution has been made in $X$ and $Z$. It is easy to see that $X_z$ and $Z_z$ are equal to zero, and that to differentiate $X$ completely with respect to $x$, it is necessary to differentiate with respect to $x$ and then to use the function of a function rule, thus $X_x + X_z(z/dx)$, and similarly for the other letters. Thus using the fact that $q = 0$, we may write the equations (5) in the form

\[
\begin{aligned}
(16) \quad & \left( X_p + \frac{\partial z}{\partial p} X_z \right) (Z_x + pZ_z) - \left( Z_p + \frac{\partial z}{\partial p} Z_z \right) (X_x + pX_z) = 0, \\
& \left( P_p + \frac{\partial z}{\partial p} P_z \right) (X_x + pX_z) - \left( X_p + \frac{\partial z}{\partial p} X_z \right) (P_x + pP_z) = 0, \\
& \left( P_p + \frac{\partial z}{\partial p} P_z \right) (Z_x + pZ_z) - \left( Z_p + \frac{\partial z}{\partial p} Z_z \right) (P_x + pP_z) = 0.
\end{aligned}
\]

It is very easy to see that these equations are now the expanded form of the determinants of the matrix (15). Hence the theorem is proved.

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INTEGRO-DIFFERENTIAL INVARIANTS OF
ONE-PARAMETER GROUPS OF
FREDHOLM TRANSFORMATIONS*

BY A. D. MICHAL

1. Statement of the Problem. The author has already considered functionals of the form $f[y(x), y'(x)]$ (depending only on a function $y(x)$ and its derivative $y'(x)$ between 0 and 1) which are invariant under an arbitrary Volterra one-parameter group of continuous transformations. The

* Presented to the Society, December 1, 1923.
† Cf. Integro-differential expressions invariant under Volterra's group of transformations in a forthcoming issue of the ANNALS OF MATHEMATICS. This paper will be referred to as "I. D. I. V."

calculation of the invariants in question was effected in the case of a large class of functionals known as analytic functionals.

The purpose of this note is to consider the problem of finding analytic functionals $f[y(r_0), y'(r_0)]$ invariant under a Fredholm group of transformations

$$y_1(x) = y(x) + \int_0^1 K(x, s | a)y(s)ds,$$

where $a$ is the parameter of this continuous one-parameter group of transformations, and where $a = 0$ corresponds to the identical transformation.

We restrict ourselves to transformations (1) for which the $y'(x)$'s exist and are continuous in the interval $I: 0 \leq x \leq 1$; $K(x, s | a)$ and $\partial K/\partial x$ are continuous in $x$ and $s$ in the square $S: 0 \leq x \leq 1, 0 \leq s \leq 1$; and $\partial K/\partial x$ is not identically zero when $a \neq 0$.

The infinitesimal transformation corresponding to (1) will be of the form

$$\delta y(x) = \left[ \int_0^1 H(x, s)y(s)ds \right] \delta a$$

with

$$\delta y'(x) = \left[ \int_0^1 H_1(x, s)y(s)ds \right] \delta a,$$

where

$$H_1(x, s) = \frac{\partial H(x, s)}{\partial x},$$

as the extended group of infinitesimal transformations.

Here follow the well known relations* between the kernel $K(x, s | a)$ of the Fredholm finite transformation (1) and the kernel $H(x, s)$ of the corresponding infinitesimal transformation (2):

* Gerhard Kowalewski, Über Funktionenräume, Wiener Sitzungsberichte, 1911, vol. 120, II A.
(4) \[ K(x, s \mid a) = \sum_{i=1}^{\infty} \frac{a^i H^i}{i!}, \]

(5) \[ H(x, s) = \left[ \frac{\partial K(x, s \mid a)}{\partial a} \right]_{a=0} = \frac{1}{a} \sum_{i=1}^{\infty} (-1)^{i-1} \frac{K^i}{i}, \quad a \neq 0, \]

where \( H^i \) and \( K^i \) are to be interpreted according to Volterra's symbolic multiplication.

By methods similar to those employed in proving the lemma of Part I of \( I. D. I. V. \), we can prove* without difficulty the following lemma.

**Lemma.** Necessary and sufficient conditions that \( H(x, s) \) and \( \partial H/\partial x \) be continuous in \( x \) and \( s \) and that \( \partial H/\partial x \) be not identically zero are that \( K(x, s \mid a) \) and \( \partial K/\partial x \) be continuous in \( x \) and \( s \) and that \( \partial K/\partial x \) be not identically zero, when \( a \neq 0 \).

2. A Sufficient Condition for Invariance.

**Theorem 1.** Let \( f[y(r_0), y'(r_0)] \) be an analytic functional of \( y(x) \) and \( y'(x) \), i.e., developable in a Volterra expansion†

\[
f_0 + \sum_{j=1}^{\infty} \frac{1}{j!} \int_0^1 \cdots \int_0^1 \sum_{k=0}^{\infty} \frac{1}{k!} f_{j-k,k}(t_1, \ldots, t_{j-k}; t_{j-k+1}, \ldots, t_j) \times \prod_{i=1}^{j-k} y(t_i) \prod_{i=j-k+1}^{j} y'(t_i) dt_1 dt_2 \cdots dt_j.
\]

We shall assume that \( f_{j-k,k} \) is continuous in its \( j \) arguments, symmetric separately in the sets of arguments \( t_1, t_2, \ldots, t_{j-k} \) and \( t_{j-k+1}, \ldots, t_j \) respectively; and for convenience we assume also that

\[
|f_{j-k,k}| < \gamma, \quad |y| < \varrho_1, \quad |y'| < \varrho_2,
\]

where \( \gamma, \varrho_1, \varrho_2 \) are positive constants. Then a sufficient condition that \( f[y(r_0), y'(r_0)] \) be invariant under a given group of transformations (1) is that it satisfy the relation

* The proof comes by a direct calculation of the series involved.
† This is a generalization of Taylor's series given by Volterra. See for example his *Leçons sur les Équations Intégrales*, 1915.
The necessary and sufficient condition that \( f[y(r_0), y'(r_0)] \) be invariant under (1) is that under (2)

\[
\delta f[y(r_0), y'(r_0)] = 0 \text{ in } y \text{ and } y'.
\]

Since the analyticity of our functionals insures the validity of a Volterra variation, we may use Volterra's* form of the variation of a functional. Then condition (9) becomes

\[
\int_0^1 H(t_i, t_i + k)
\int_0^1 H(t, t_i + k) f_i, k(t_1, \ldots, t_i; t_i + 1, \ldots, t_i + k - 1, t) dt
\]

\[
- \int_0^1 H(t, t_i + k) f_{i+1, -1}(t, t_1, \ldots, t_i; t_i + 1, \ldots, t_i + k - 1) dt.
\]

Substituting in (10) the values of \( \delta y(t) \) and \( \delta y'(t) \) as given by (2) and (3), respectively, rearranging and dividing through by \( da \), we get

\[
\int_0^1 y(s) \left[ \int_0^1 f'y(t)H(t, s) dt + \int_0^1 f''y(t)H_1(t, s) ds \right] dt = 0
\]

in \( y \). We may now apply Lemma 2 of I. D. I. V.; doing so, we find

\[
\int_0^1 f''y(t)H_1(t, s) dt = -\int_0^1 f'y(t)H(t, s) dt.
\]

Such operations as functional differentiations term by term are valid since the series involved are uniformly convergent under our hypotheses.† Calculating the partial functional derivatives \( f_y(t) \) and \( f_{y'}(t) \), respectively, and substituting them in (12), we get by an easy reduction‡

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* A more general expression for \( \delta f \) would be in the form of Stieltjes integrals.


‡ Cf. similar reductions of I. D. I. V.
\[
\sum_{j=1}^{\infty} \left( \frac{1}{(j-1)!} \right) \int_{0}^{1} \int_{0}^{1} \cdots \int_{0}^{1} \left[ \sum_{l=0}^{j-1} \binom{j-1}{l} H_{l}(t, s)f_{j-l-i, \ t-1}(t_{1}, \ldots, t_{j-l}, \ t) \right] dt_{1} \cdots dt_{j-1} dt
\]

(13)

\[
\equiv -\sum_{j=1}^{\infty} \left( \frac{1}{(j-1)!} \right) \int_{0}^{1} \int_{0}^{1} \cdots \int_{0}^{1} \left[ \sum_{k=0}^{j-1} \binom{j-1}{k} H(t, s)f_{j-k-k}(t, t_{1}, \ldots, t_{j-1}) \right] dt_{1} \cdots dt_{j-1} dt,
\]

where \( k = l + 1 \). Equating coefficients of similar terms in \( y \) and \( y' \), we find

\[
\int_{0}^{1} H_{l}(t, s)f_{j-l-i, \ t-1}(t_{1}, \ldots, t_{j-l}, \ t) dt \equiv -\int_{0}^{1} H(t, s)f_{j-l-i}(t, t_{1}, \ldots, t_{j-1}) dt,
\]

which can be written in the form (8).

3. Calculation of the Invariants \( f[y(t_{0}), y'(t_{0})] \). In order that \( f[y(t_{0}), y'(t_{0})] \) be invariant under (1) it is sufficient that the following recurrence formula hold

(14)

\[
f_{i, k}(t_{1}, \ldots, t_{i}; t_{i+1}, \ldots, t_{i+k-1}, t)
\]

\[
\equiv -\frac{H(t, t_{i+k})}{H_{1}(t, t_{i+k})} f_{i+1, k-1}(t, t_{1}, \ldots, t_{i}; t_{i+1}, \ldots, t_{i+k-1}).
\]

We shall now prove the following theorem.

**Theorem II.** A necessary and sufficient condition on (1) that an analytic functional \( f[y(t_{0}), y'(t_{0})] \) be invariant under (1) when (14) holds is that the kernel \( H(x, s) \) of the infinitesimal transformation be of the form

(15)

\[
H(x, s) \equiv \psi(s)e^{\gamma c},
\]
where \( \psi(s) \) is an arbitrary function of \( s \), and where \( c \) is a constant.*

It is evident from (14) that

\[
\frac{H(t, t_{i+k})}{H_1(t, t_{i+k})}
\]

must be independent of \( t_{i+k} \), and hence it is necessary that it be a function of \( t \) alone, say \( \varphi(t) \). On applying (14) until \( f_{i,k} \) is written in terms of \( f \)'s with second index zero, we get the recurrence formula

\[
(16) \quad f_{i,k}(t_1, t_2, \ldots, t_i; t_i+1, \ldots, t_i+k-1, t) = (-1)^i \varphi(t)^i f_{i+k,0}(t, t_1, \ldots, t_i, t_i+1, \ldots, t_i+k-1).
\]

By hypothesis \( f_{i+k,0} \) is symmetric in all its arguments. Therefore, interchanging \( t_1 \) and \( t_{i+1} \) leaves the right-hand side of (16) unchanged. Hence if (16) is to hold, \( f_{i,k} \) must be symmetric with respect to \( t_1 \) and \( t_{i+1} \), and therefore it must be symmetric in all its arguments. On interchanging \( t \) and any \( t_j \) in (16), we see at once that \( \varphi(t) \) must be a constant, say \( c \); i.e., \( H(x, s) \) must satisfy the equation

\[
(17) \quad c \frac{\partial H(x, s)}{\partial x} - H(x, s) = 0,
\]

whose most general solution is (15).

We may now remark that the arbitrariness of the coefficients \( f_{i+k,0} \), in terms of which all the other \( f_{i,k} \)'s can be evaluated, on making use of the recurrence formula, enables us to state immediately the following theorem.

**Theorem III.** Let the kernel \( H(x, s) \) of the infinitesimal transformation (2) be of the form \( \psi(s)e^{x/s} \), and let us take an analytic functional \( f[y(x_0), y'(x_0)] \), all of whose \( f_{i,k} \)'s are symmetric in all their arguments, and assign arbitrarily for initial conditions the coefficients \( f_{i+k,0} \) in its Volterra expansion; that is, take an arbitrary \( F[y(x_0)] \) such that \( F[y(x_0)] = f[y(x_0), y'(x_0)] \), and for convenience take \( y_0(x) = 0 \). Then, if the \( f_{i,k} \)'s are calculated by the recurrence formula

\[
* \quad \text{That is, if } H(x, s) = \psi(s)e^{x/s}, \text{ we may assert that invariant analytic functionals } f[y(x_0), y'(x_0)] \text{ always exist.}
we shall have an analytic functional \( f[y(\tau'_0), y'(\tau'_0)] \) which will be invariant under a transformation (1) whose kernel \( K(x, s \mid a) \) is given by

\[
K(x, s \mid a) = \sum_{i=1}^{\infty} \frac{a^i}{i!} \psi(s)e^{\alpha/s} \int_0^1 \int_{0(t-1)}^{\tau_1} \cdots \int_{0(t-1)}^{\tau_1} e^{i \sum_{l=1}^{t-1} \psi(t_l)dt_l} \cdots dt_{t-1}.
\]

4. Example. We here give an easy example in which the direct verification by means of the finite transformation is very simple. Let us suppose that

\[
\delta y(x) = \left[ e^x \int_0^1 sy(s)ds \right] \delta a
\]

is the given infinitesimal transformation, i.e., that \( H(x, s) = se^x \). By means of an easy calculation, the finite transformation may be written in the form

\[
y(x) = y(x) + (e^a - 1)e^x \int_0^1 sy(s)ds,
\]

i.e., \( K(x, s \mid a) = (e^a - 1)e^x s \). Let us take for initial condition

\[
f[y(\tau'_0), 0] \equiv F[y(\tau'_0)]
\]

\[
= f_{00} + \int_0^{\tau_1} f_{10}(t_1)y(t_1)dt_1 + \frac{1}{2!} \int_0^{\tau_1} dt_2 \int_0^{\tau_1} f_{20}(t_1, t_2)y(t_1)y(t_2)dt_1.
\]

Then the functional \( f[y(\tau'_0), y'(\tau'_0)] \) given by

\[
f[y(\tau'_0), y'(\tau'_0)] = f_{00} + \int_0^{\tau_1} f_{10}(t_1)[y(t_1) - y'(t_1)]dt_1 + \frac{1}{2!} \int_0^{\tau_1} dt_2 \int_0^{\tau_1} f_{20}(t_1, t_2)[y(t_1)y(t_2) - 2y(t_1)y'(t_2) + y'(t_1)y'(t_2)]dt
\]

is invariant under (20).

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