Let us see what the effect will be when the value of z as obtained from $\varrho = 0$ is substituted in equations (1). Suppose that the substitution has been made in X and Z. It is easy to see that X_z and Z_z are equal to zero, and that to differentiate X completely with respect to x, it is necessary to differentiate with respect to x and then to use the function of a function rule, thus $X_x + X_z(\partial z/\partial x)$, and similarly for the other letters. Thus using the fact that $\varrho = 0$, we may write the equations (5) in the form

$$
(16)\left\{
\begin{aligned}
&\left(X_p+\frac{\partial z}{\partial p}X_z\right)(Z_x+pZ_z)-\left(Z_p+\frac{\partial z}{\partial p}Z_z\right)(X_x+pX_z)=0,\\
&\left(P_p+\frac{\partial z}{\partial p}P_z\right)(X_x+pX_z)-\left(X_p+\frac{\partial z}{\partial p}X_z\right)(P_x+pP_z)=0,\\
&\left(P_p+\frac{\partial z}{\partial p}P_z\right)(Z_x+pZ_z)-\left(Z_p+\frac{\partial z}{\partial p}Z_z\right)(P_x+pP_z)=0.
\end{aligned}
\right.
$$

It is very easy to see that these equations are now the expanded form of the determinants of the matrix (15). Hence the theorem is proved.

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INTEGRO-DIFFERENTIAL INVARIANTS OF ONE-PARAMETER GROUPS OF FREDHOLM TRANSFORMATIONS*

BY A. D. MICHAL

1. *Statement of the Problem.* The author† has already considered functionals of the form $f[y(\tau_0'), y'(\tau_0')]$ (depending only on a function $y(\tau)$ and its derivative $y'(\tau)$ between 0 and 1) which are invariant under an arbitrary Volterra one-parameter group of continuous transformations. The

* Presented to the Society, December 1, 1923.

† Cf. *Integro-differential expressions invariant under Volterra's group of transformations* in a forthcoming issue of the ANNALS OF MATHEMATICS. This paper will be referred to as "*I. D. I. V.*"

calculation of the invariants in question was effected in the case of a large class of functionals known as analytic functionals.

The purpose of this note is to consider the problem of finding analytic functionals $f[y(\tau_0'), y'(\tau_0')]$ invariant under a Fredholm group of transformations

$$(1) \qquad y_1(x) = y(x) + \int_0^1 K(x, s \mid a) y(s) ds,$$

where a is the parameter of this continuous one-parameter group of transformations, and where $a = 0$ corresponds to the identical transformation.

We restrict ourselves to transformations (1) for which the $y'(\tau)$'s exist and are continuous in the interval $I: 0 \leqq \tau \leqq 1$; $K(x, s \mid a)$ and $\partial K / \partial x$ are continuous in x and s in the square $S: 0 \leqq x \leqq 1,\ 0 \leqq s \leqq 1$; and $\partial K / \partial x$ is not identically zero when $a \neq 0$.

The infinitesimal transformation corresponding to (1) will be of the form

$$(2) \qquad \delta y(x) = \left[\int_0^1 H(x, s) y(s) ds \right] \delta a$$

with

$$(3) \qquad \delta y'(x) = \left[\int_0^1 H_1(x, s) y(s) ds \right] \delta a,$$

where

$$H_1(x, s) \equiv \frac{\partial H(x, s)}{\partial x},$$

as the extended group of infinitesimal transformations.

Here follow the well known relations* between the kernel $K(x, s \mid a)$ of the Fredholm finite transformation (1) and the kernel $H(x, s)$ of the corresponding infinitesimal transformation (2):

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* Gerhard Kowalewski, *Über Funktionenräume,* WIENER SITZUNGSBERICHTE, 1911, vol. 120, IIA.

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$$K(x, s \mid a) = \sum_{i=1}^{\infty} \frac{a^i H^i}{i!}, \tag{4}$$

$$H(x, s) = \left[\frac{\partial K(x, s \mid a)}{\partial a}\right]_{a=0} = \frac{1}{a}\sum_{i=1}^{\infty}(-1)^{i-1}\frac{K^i}{i}, \quad a \neq 0, \tag{5}$$

where H^i and K^i are to be interpreted according to Volterra's symbolic multiplication.

By methods similar to those employed in proving the lemma of Part I of *I. D. I. V.*, we can prove* without difficulty the following lemma.

LEMMA. *Necessary and sufficient conditions that $H(x, s)$ and $\partial H/\partial x$ be continuous in x and s and that $\partial H/\partial x$ be not identically zero are that $K(x, s \mid a)$ and $\partial K/\partial x$ be continuous in x and s and that $\partial K/\partial x$ be not identically zero, when $a \neq 0$.*

2. *A Sufficient Condition for Invariance.*

THEOREM 1. *Let $f[y(\tau_0'), y'(\tau_0')]$ be an analytic functional of $y(\tau)$ and $y'(\tau)$,* i. e., *developable in a Volterra expansion*†

$$f_{00} + \sum_{j=1}^{\infty}\left\{\frac{1}{j!}\int_0^1\int_0^1\cdots\int_0^1\left[\sum_{k=0}^{j}\binom{j}{k} f_{j-k,k}(t_1, \cdots, t_{j-k}; t_{j-k+1}, \cdots, t_j)\right.\right.$$
$$\left.\left.\times\prod_{i=1}^{j-k} y(t_i)\prod_{i=j-k+1}^{j} y'(t_i)\right] dt_1 dt_2 \cdots dt_j\right\}. \tag{6}$$

We shall assume that $f_{j-k,k}$ is continuous in its j arguments, symmetric separately in the sets of arguments $t_1, t_2, \ldots, t_{j-k}$ and $t_{j-k+1}, \ldots, t_j$ respectively; *and for convenience we assume also that*

$$|f_{j-k,k}| < \gamma, \qquad |y| < \varrho_1, \qquad |y'| < \varrho_2, \tag{7}$$

where γ, ϱ_1, ϱ_2 are positive constants. Then a sufficient condition that $f[y(\tau_0'), y'(\tau^0)]$ be invariant under a given group of transformations (1) *is that it satisfy the relation*

* The proof comes by a direct calculation of the series involved.

† This is a generalization of Taylor's series given by Volterra. See for example his *Leçons sur les Équations Intégrales*, 1913.

$$(8)\quad \int_0^1 H_1(t,\, t_{i+k}) f_{i,\,k}(t_1,\, \ldots,\, t_i;\, t_{i+1},\, \ldots,\, t_{i+k-1},\, t)dt$$
$$-\equiv \int_0^1 H(t,\, t_{i+k}) f_{i+1,\,k-1}(t,\, t_1,\, \ldots,\, t_i;\, t_{i+1},\, \ldots,\, t_{i+k-1})dt.$$

The necessary and sufficient condition that $f[y(\tau_0'),\, y'(\tau_0)]$ be invariant under (1) is that under (2)

$$(9)\qquad \delta f[y(\tau_0'),\, y'(\tau_0')] \equiv 0 \text{ in } y \text{ and } y'.$$

Since the analyticity of our functionals insures the validity of a Volterra variation, we may use Volterra's* form of the variation of a functional. Then condition (9) becomes

$$(10)\qquad \int_0^1 f_y(t)\delta y(t)dt + \int_0^1 f_{y'}(t)\delta y'(t)dt \equiv 0$$

in y and y', where $f_y(t)$ and $f_{y'}(t)$ are the partial functional derivatives of $f[y(\tau_0'),\, y'(\tau_0')]$ with respect to $y(\tau)$ and $y'(\tau)$, respectively, both taken at the point t.

Substituting in (10) the values of $\delta y(t)$ and $\delta y'(t)$ as given by (2) and (3), respectively, rearranging and dividing through by δa, we get

$$(11)\quad \int_0^1 y(s)\Big[\int_0^1 f_y(t)H(t,\, s)dt + \int_0^1 f_{y'}(t)H_1(t,\, s)dt\Big]ds \equiv 0$$

in y. We may now apply Lemma 2 of *I. D. I. V.*; doing so, we find

$$(12)\qquad \int_0^1 f_{y'}(t)H_1(t,\, s)dt = -\int_0^1 f_y(t)H(t,\, s)dt.$$

Such operations as functional differentiations term by term are valid since the series involved are uniformly convergent under our hypotheses.† Calculating the partial functional derivatives $f_y(t)$ and $f_{y'}(t)$, respectively, and substituting them in (12), we get by an easy reduction‡

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* A more general expression for δf would be in the form of Stieltjes integrals.

† Cf. *I. D. I. V.*, Part II, and Volterra's *Leçons sur les Équations Intégrales,* 1913, p. 18.

‡ Cf. similar reductions of *I. D. I. V.*

$$\sum_{j=1}^{\infty}\left\{\frac{1}{(j-1)!}\int_0^1\int_{0_{(j)}}^1\cdots\int_0^1\left[\sum_{l=0}^{j-1}\binom{j-1}{l}H_1(t, s)f_{j-1-l,\, l+1}(t_1, \dots\right.\right.$$

$$\dots, t_{j-1-l};\ t_{j-l}, \dots, t_{j-1}, t)$$

$$\left.\left.\times\prod_{i=1}^{j-1-l} y(t_i)\prod_{i=j-l}^{j-1} y'(t_i)\right]dt_1 \dots dt_{j-1}dt\right\}$$

(13)

$$\equiv-\sum_{j=1}^{\infty}\left\{\frac{1}{(j-1)!}\int_0^1\int_{0_{(j)}}^1\cdots\int_0^1\left[\sum_{k=0}^{j-1}\binom{j-1}{k}H(t,s)f_{j-k,k}(t, t_1, \dots\right.\right.$$

$$\dots, t_{j-k-1};\ t_{j-k}, \dots, t_{j-1})$$

$$\left.\left.\times\prod_{i=1}^{j-k-1} y(t_i)\prod_{i=j-k}^{j-1} y'(t_i)\right]dt_1 \dots dt_{j-1}dt\right\},$$

where $k=l+1$. Equating coefficients of similar terms in y and y', we find

$$\int_0^1 H_1(t, s)f_{j-1-l,\, l+1}(t_1, \dots, t_{j-1-l};\ t_{j-l}, \dots, t_{j-1}, t)dt$$

$$\equiv-\int_0^1 H(t, s)f_{j-l,\, l}(t, t_1, \dots, t_{j-1-l};\ t_{j-l}, \dots, t_{j-1})dt,$$

which can be written in the form (8).

3. *Calculation of the Invariants* $f[y(\tau_0'), y'(\tau_0')]$. In order that $f[y(\tau_0'), y'(\tau_0')]$ be invariant under (1) it is sufficient that the following recurrence formula hold

$$f_{i,k}(t_1, \dots, t_i;\ t_{i+1}, \dots, t_{i+k-1}, t) \tag{14}$$

$$\equiv-\frac{H(t, t_{i+k})}{H_1(t, t_{i+k})}f_{i+1,\, k-1}(t, t_1. \dots, t_i;\ t_{i+1}, \dots, t_{i+k-1}).$$

We shall now prove the following theorem.

THEOREM II. *A necessary and sufficient condition on* (1) *that an analytic functional* $f[y(\tau_0), y'(\tau_0')]$ *be invariant under* (1) *when* (14) *holds is that the kernel* $H(x, s)$ *of the infinitesimal transformation be of the form*

$$H(x, s)\equiv\psi(s)e^{x/c}, \tag{15}$$

*where $\psi(s)$ is an arbitrary function of s, and where c is a constant.**

It is evident from (14) that

$$\frac{H(t,\ t_{i+k})}{H_1(t,\ t_{i+k})}$$

must be independent of t_{i+k}, and hence it is necessary that it be a function of t alone, say $\varphi(t)$. On applying (14) until $f_{i,k}$ is written in terms of f's with second index zero, we get the recurrence formula

$$\begin{aligned}(16)\qquad & f_{i,k}(t_1,\ t_2,\ \ldots,\ t_i;\ t_{i+1},\ \ldots,\ t_{i+k-1},\ t)\\ &= (-1)^k[\varphi(t)]^k f_{i+k,0}(t,\ t_1,\ \ldots,\ t_i,\ t_{i+1},\ \ldots,\ t_{i+k-1}).\end{aligned}$$

By hypothesis $f_{i+k,0}$ is symmetric in all its arguments. Therefore, interchanging t_1 and t_{i+1} leaves the right-hand side of (16) unchanged. Hence if (16) is to hold, $f_{i,k}$ must be symmetric with respect to t_1 and t_{i+1}, and therefore it must be symmetric in all its arguments. On interchanging t and any t_j in (16), we see at once that $\varphi(t)$ must be a constant, say c; i. e., $H(x,s)$ must satisfy the equation

$$(17)\qquad c\frac{\partial H(x,s)}{\partial x} - H(x,s) = 0,$$

whose most general solution is (15).

We may now remark that the arbitrariness of the coefficients $f_{i+k,0}$, in terms of which all the other $f_{i,k}$'s can be evaluated, on making use of the recurrence formula, enables us to state immediately the following theorem.

THEOREM III. *Let the kernel $H(x,s)$ of the infinitesimal transformation* (2) *be of the form $\psi(s)e^{x/c}$, and let us take an analytic functional $f[y(\tau_0'),\ y'(\tau_0')]$, all of whose $f_{i,k}$'s are symmetric in all their arguments, and assign arbitrarily for initial conditions the coefficients $f_{i+k,0}$ in its Volterra expansion; that is, take an arbitrary $F[y(\tau_0')]$ such that $F[y(\tau_0')] \equiv f[y(\tau_0'),\ y_0'(\tau_0')]$, and for convenience take $y_0'(\tau) \equiv 0$. Then, if the $f_{i,k}$'s are calculated by the recurrence formula*

* That is, if $H(x,s) \equiv \psi(s)e^{x/c}$, we may assert that invariant analytic functionals $f[y(\tau_0')\ y'(\tau_0')]$ always exist.

(18) $f_{i,k}(t_1, \ldots, t_{i+k}) = (-1)^k c^k f_{i+k,0}(t_1, \ldots, t_{i+k})$,

we shall have an analytic functional $f[y(\tau_0'), y'(\tau_0)]$ *which will be invariant under a transformation* (1) *whose kernel* $K(x, s \mid a)$ *is given by*

$$K(x, s \mid a)$$

$$(19)\quad = \sum_{i=1}^{\infty} \frac{a^i}{i!} \psi(s) e^{x/c} \int_0^1 \int_0^1 \cdots \int_0^1 e^{\frac{1}{c}\sum_{j=1}^{i-1} t_j} \prod_{l=1}^{i-1} \psi(t_l) dt_1 \ldots dt_{i-1}.$$

4. *Example.* We here give an easy example in which the direct verification by means of the finite transformation is very simple. Let us suppose that

$$\delta y(x) = \left[e^x \int_0^1 s y(s) ds \right] \delta a$$

is the given infinitesimal transformation, i. e., that $H(x, s) = se^x$. By means of an easy calculation, the finite transformation may be written in the form

$$(20)\qquad y_1(x) = y(x) + (e^a - 1) e^x \int_0^1 s y(s) ds,$$

i. e., $K(x, s \mid a) = (e^a - 1) e^x s$. Let us take for initial condition

$$f[y(\tau_0'), 0] \equiv F[y(\tau_0')]$$

$$\equiv f_{00} + \int_0^1 f_{10}(t_1) y(t_1) dt_1 + \frac{1}{2!} \int_0^1 dt_2 \int_0^1 f_{20}(t_1, t_2) y(t_1) y(t_2) dt_1.$$

Then the functional $f[y(\tau_0'), y'(\tau_0')]$ given by

$$f[y(\tau_0'), y'(\tau_0')] = f_{00} + \int_0^1 f_{10}(t_1)[y(t_1) - y'(t_1)] dt_1$$

$$+ \frac{1}{2!} \left[\int_0^1 dt_2 \int_0^1 f_{20}(t_1, t_2) \{ y(t_1) y(t_2) - 2y(t_1) y'(t_2) + y'(t_1) y'(t_2) \} \right] dt$$

is invariant under (20).

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