relations are

\[(3) \quad h_1^{d_1} = 1, \quad h_2^{d_2} = 1, \quad \ldots, \quad h_r^{d_r} = 1\]

where those of the exponents \(d_1, d_2, \ldots, d_r\) which are not 1 are the invariant factors of the matrix \(|y_u|\).

An operation of \(\tilde{G}\) is of finite period if and only if it is in the group \(H\) generated by the operations \(h_1, h_2, \ldots, h_r\) and the relations (3). Hence \(H\) contains all operations of \(\tilde{G}\) of finite period. Moreover \(H\) is a finite group because it is commutative and generated by a finite number of operations each of finite period. The invariant factors of \(|y_u|\) are invariants of \(H\) (Kronecker, BERLINER MONATSBERICHTE, 1870, p. 885).

These invariant factors are what Tietze calls the Poincaré numbers of the group \(G\). They are invariants of \(G\) because \(G\) determines the commutative group \(\tilde{G}\) uniquely and \(\tilde{G}\) determines the finite group \(H\) uniquely and \(H\) determines the invariant factors uniquely.

PRINCETON UNIVERSITY

---

**ANalytic Functions AND PERIODICITY***

BY J. F. RITT

This paper presents two theorems which show that the condition that a function be periodic can be analyzed, in different ways, into a set of requirements, from the satisfaction of only one of which, if the function is analytic, the periodicity of the function can be inferred. The theorems are

**Theorem A.** If \(f(z)\) is a uniform analytic function, and if an \(a > 0\) exists such that every \(z_1\) at which \(f(z)\) is analytic is the center of a circle of radius \(a\) on which a \(z_2\) lies at which \(f(z)\) is analytic and assumes the same value as at \(z_1\), then \(f(z)\) is periodic, and has a period of modulus \(a\).

* Presented to the Society May 3, 1924.
Theorem B. If \( f(z) \) is a uniform analytic function, and if every \( z_1 \) at which \( f(z) \) is analytic belongs to a sequence of non-coincident points in arithmetic progression, at each of which \( f(z) \) is analytic and assumes the same value as at \( z_1 \), then \( f(z) \) is periodic.

In A, the condition that \( z_1 - z_2 \) have a constant modulus can be replaced by the condition that it have a constant argument, or, for that matter, a constant real or pure imaginary part. Also, in both theorems, the condition that \( f(z) \) be uniform is not essential, and is made only to keep cumbersome statements out of the proof.

Theorems similar to the above can be proved for other peculiarities than periodicity. Many types of functional equations can be studied from the point of view of this paper.

We shall use the following lemma.

Lemma. Given a non-constant uniform analytic function \( F(z) \), and a transformation which carries every \( z_1 \) at which \( f(z) \) is analytic into a \( z_2 \) at which \( f(z) \) is analytic and assumes the same value as at \( z_1 \), there exists an analytic function \( \psi(z) \) such that \( z_2 = \psi(z_1) \) for a set of points \( z_1 \) dense in some area. For every \( z \) in this area, \( f(\psi(z)) = f(z) \).

If \( f(z) \) is analytic at \( z \) and if \( f'(z) \neq 0 \), there are circles with \( z \) as center inside of which \( f(z) \) is analytic and assumes no value more than once. For any \( z \), one of these circles—call it \( c \)—has a maximum (perhaps infinite) radius. Those maximum circles \( c \) for the center of which, \( z = x + iy \), both \( x \) and \( y \) are rational, are countable. Let them be arranged in a sequence

\[ c_1, c_2, \ldots, c_n, \ldots \]

It is easy to see that every \( z \) for which \( f'(z) \) is analytic and \( f'(z) \neq 0 \) is interior to some \( c_n \).

Take now any circle \( \gamma \) inside of which \( f(z) \) is analytic and assumes no value more than once. For every \( z_1 \) in \( \gamma \), the hypothesis of the lemma provides a \( z_2 \). Those points \( z_2 \) at which \( f'(z_2) = 0 \) are countable, for the zeros of an
analytic function which is not identically zero are isolated. Every other \( z_2 \) lies within some \( c_n \); associate with its \( z_1 \) the \( c_n \) of smallest \( n \) in which \( z_2 \) lies. In this way those points \( z_1 \) of \( \gamma \) for which \( f'(z_2) \neq 0 \) are separated into a countable number of sets. Not all of these sets can be nowhere dense, for their sum differs from all of \( \gamma \) by a countable set.

Hence there is a \( c_n \) and a set of points—call it \( M \)—dense in some area within \( \gamma \), such that for any \( z_1 \) of \( M \), \( z_2 \) lies within \( c_n \).

Take, and hold fast momentarily, any \( z_1 \) of \( M \) which is the center of some circle—call it \( I \)—in which \( M \) is dense. As \( f(z_1) = f(z_2) \), we can take \( I \) so small that \( f(z) \) assumes no value in \( I \) which it does not assume within \( c_n \). To associate with every point of \( I \) that point of \( c_n \) at which \( f(z) \) assumes the same value is to define a \( \varphi(z) \) which is analytic in \( I \). Furthermore, for a set of points \( z_1 \) dense in \( I \), \( z_2 = \varphi(z_1) \). This proves the lemma.

It is now easy to prove Theorem A. Because \( |z_2 - z_1| \) is constant, the modulus of \( \varphi(z) \) is constant for a set of points dense in \( I \). As the modulus of \( \varphi(z) \) is continuous, it is constant throughout \( I \). An analytic function with a constant modulus is a constant. Let \( h \) be the constant value of \( \varphi(z) \); of course \( h \neq 0 \). Then \( f(z + h) = f(z) \) for every \( z \) in \( I \), and hence throughout the domain of existence of \( f(z) \).

Considering Theorem B, suppose that, for every \( z_1 \) at which \( f(z) \) is analytic, a number \( h(z_1) \neq 0 \) exists such that \( f(z) \) is analytic, and equal to \( f(z_1) \), at every \( z_1 + mh(z_1) \) \((m = 1, 2, \ldots)\).

According to the lemma, there exists a circle \( I \), a set \( M \) dense in \( I \), and an analytic \( \varphi(z) \), such that \( \varphi(z_1) = z_1 + h(z_1) \) for every \( z_1 \) in \( M \).

Suppose that \( \varphi(z) - z \) is not constant. Barring out the trivial case in which \( f(z) \) is constant, we may assume that neither \( f(z) \) nor \( \varphi(z) - z \) assumes any value more than once in \( I \); this can always be brought about by replacing \( I \) by some circle within \( I \). Let \( I' \) and \( I'' \) be two circles
concentric with \( \Gamma \), and of radii respectively one-half and one-quarter that of \( \Gamma \).

Consider the function

\[
\frac{z}{m} + [\varphi(z) - z],
\]

where \( m \) is any positive integer. As \( m \) increases this function approaches \( \varphi(z) - z \) uniformly in \( \Gamma \), so that, for \( m \) large enough, it assumes no value more than once in \( \Gamma' \), but assumes in \( \Gamma' \) every value which \( \varphi(z) - z \) assumes in \( \Gamma'' \). The values assumed in \( \Gamma'' \) certainly cover some circle \( \Delta \).

Consequently, for \( m \) large enough, the function

\[
\psi_m(z) = z + m[\varphi(z) - z]
\]

maps \( \Gamma' \) conformally, in a one-to-one manner, upon a region \( \Sigma_m \) which covers a circle \( \Delta_m \), of radius \( m \) times that of \( \Delta \).

Let \( z \) be in \( \Gamma' \). Associate with \( \psi_m(z) \) the number \( f(z) \). This defines a \( g_m(z) \), analytic in \( \Sigma_m \). Now \( \psi_m(z) \) maps those points of \( M \) which lie in \( \Gamma' \) into a set of points dense in \( \Sigma_m \). Since \( \psi_m(z_1) = z_1 + mh(z_1) \) for \( z_1 \) in \( M \), \( f(z) \) and \( g_m(z) \) are equal for a set of points dense in \( \Sigma_m \), and therefore throughout \( \Sigma_m \).

As \( f(z) \) assumes no value more than once in \( \Gamma' \), \( f(z) \) assumes no value more than once in \( \Sigma_m \). Consider any \( z_1 \) of \( M \) such that \( \psi_m(z_1) \) is very close to the center of \( \Delta_m \). Since \( h(z) = \varphi(z) - z \) for \( z \) in \( M \), \( h(z) \) is bounded for \( z \) at once in \( M \) and in \( \Gamma' \). Hence if \( m \) is large, so that \( \Delta_m \) is large, \( \psi_m(z_1) + h(z_1) \), at which \( f(z) \) takes the same value as at \( \psi_m(z_1) \), will lie in \( \Delta_m \), and so in \( \Sigma_m \). This contradiction with the statement at the head of the paragraph proves that \( \varphi(z) - z \) is constant; it is not zero, for no \( h(z) \) is zero. As \( f[\varphi(z)] = f(z) \) for \( z \) in \( \Gamma \), \( f(z) \) is periodic.

\textit{Columbia University}