THE WESTERN CHRISTMAS MEETING OF THE SOCIETY

The twenty-second Western meeting of the Society was held on Friday, December 26, 1924, at the University of Chicago. The first session opened at 10 a. m. in Room 32, Ryerson Laboratory.

The attendance at this meeting numbered about sixty, including the following fifty-two members of the Society:

F. E. Allen, Barnard, Barnett, Bliss, O. E. Brown, Carmichael, Chapelon, H. H. Conwell, H. T. Davis, Dickson, Dostal, Dresden, Edington, Garabedian, Gouwens, G. H. Graves, V. G. Grove, Hildebrandt, Hodge, Hotelling, Louis Ingold, Ingraham, Isaacs, Kempner, Kinney, Kouperman, E. P. Lane, Latimer, Laves, Libman, March, G. A. Miller, Miser, E. H. Moore, E. J. Moulton, C. I. Palmer, Plapp, Rider, Roth, Rowland, Schottenfels, Shaw, Shohat, Simmons, Slaught, E. L. Thompson, J. S. Turner, Van Vleck, Vass, M. B. White, Frederick Wood, F. E. Wood.

The committee appointed to arrange for a symposium at the meeting in April 1925 reported that the symposium lecture, on "Some of the mathematical aspects of cosmology", would be given by Professor W. D. MacMillan.

At the dinner held on Friday evening, at the Del Prado Hotel, 34 persons were present.

Professor Hildebrandt, Vice-President of the Society, presided over the sessions, at which the following papers were presented. The papers of Professors MacNeish, Wahlin, Betz, Dodd and Ettlinger were read by title.

1. Professor F. E. Wood: *Sense relations between the pairs of corresponding triangles of a Desargues configuration.*

In this paper an investigation is made of the relative senses of the triangles in each of the ten pairs of perspective triangles of a Desargues configuration, starting with axioms and theorems developed in the foundations of geometry. A unique canonical lettering for a general Desargues configuration is set up and seven types of Desargues configurations with the canonical lettering are discussed. It is proved that the triangles of each of the ten pairs will have the same sense if and only if each

vertex is outside both the triangles corresponding to it, that the senses of the triangles of each of four and only four pairs are the same if and only if there is a vertex inside both its corresponding triangles, while if neither of the above two conditions is satisfied then the senses of the triangles of each of six and only six pairs are the same.

2. Professor E. P. Lane: *Bundles and pencils of nets on a surface.*

A bundle of nets on a surface is the class of nets such that the two component families of curves of each can be taken in such an order that at every surface point its tangents form with the tangents of a fundamental net the same cross ratio. The class of conjugate nets is a bundle for which the fundamental net is the asymptotic net and the constant cross ratio is -1. A pencil of nets in a bundle is a one-parameter family of nets in the bundle such that at every surface point the two tangents of each net form with the tangents of a fundamental net of the bundle a constant cross ratio.

This paper develops a general theory of bundles and pencils of nets. The theory finds application in a critical examination of the existing theory of conjugate nets; it shows what features of this theory are due to the constancy of the cross ratio and what features are due to the size of this constant. In these studies the importance of the ray-conic is again emphasized.

3. Professor V. G. Grove: *A theory of a general net on a surface.*

In this paper, the author sets up a complete system of invariants and covariants of the differential equations arising in the study of a general net. Necessary and sufficient conditions that a net be identically self dual are established. An extended study of Green's relation R between two congruences is made. There exists one and only one pair of congruences in relation R with respect to a general net, whose lines are at the same time reciprocal polars with respect to both quadrics osculating the parametric ruled surfaces of tangents. The associate conjugate net of a general net is discussed. It is shown that many of the properties of the associate conjugate net of a conjugate net hold when the net is general.

4. Mr. J. A. Shohat: *On the polynomial of the best approximation to a given continuous function.*

Consider two functions $f(x)$ and $p(x)$ defined on a finite interval (a, b); $f(x)$ is bounded and integrable; $p(x)$ is integrable and not negative. Assuming the existence of two numbers α, β such that $\int_{x_1}^{x_2} p(x)\,dx > 0$ $(\alpha \leqq x_1 < x_2 \leqq \beta$; $a \leqq \alpha$; $\beta \leqq b)$, the author proves the existence and the uniqueness of a polynomial $U_{n,k}(x)$ of degree $\leqq n$, minimizing the integral $\int_a^b p(x)\,|f(x) - U_{n,k}(x)|^k\,dx$, for any $k \geqq 1$.

Furthermore, if $f(x)$ is continuous and $\int_\alpha^\beta p(x)\,dx > 0$ for any α, β belonging to (a, b), then the said minimizing polynomial tends for $k \to \infty$ uniformly to the polynomial of the nth degree of the best approximation (in Tchebycheff's sense) in (a, b) to the function $f(x)$. Similar results hold for trigonometric approximation.

The results given above are a generalization of those obtained by G. Pólya (COMPTES RENDUS, 1913) and by D. Jackson (see the TRANSACTIONS and BULLETIN of the Society, 1921–1924).

5. Professor P. R. Rider: *The figuratrix in the calculus of variations.*

This paper considers integrals of the form $\int_{t_1}^{t_2} f(x, y, \tau)\sqrt{x'^2 + y'^2}\,dt$, $\int_{t_1}^{t_2} f(x, y, z, \tau, \sigma)\sqrt{x'^2 + y'^2 + z'^2}\,dt$, in which the arguments of f are functions of t. The quantities τ and σ are angles defined by the equations $\tan \tau = y'/x'$, $\sin \sigma = z'/\sqrt{x'^2 + y'^2 + z'^2}$. For the first integral, the figuratrix of the point (x, y) is defined as the envelope of the variable line $p \cos \tau + q \sin \tau = f(\tau)$, p and q being rectangular coordinates with reference to the point $P(x, y)$ as origin; for the second integral, the figuratrix of the point (x, y, z) is defined as the envelope of the two-parameter family of planes $p \cos \tau \cos \sigma + q \sin \tau \cos \sigma + r \sin \sigma = f(\tau, \sigma)$. The figuratrix as thus defined is proved to be identical with the figuratrix as defined by Hadamard (*Leçons sur le Calcul des Variations*, vol. I, pp. 92, 96). Attention is called to a number of the interesting properties of the figuratrix which enable one to interpret geometrically some of the well known functions and theorems of the calculus of variations. The results are generalized to the case of n dimensions.

6. Dr. H. F. MacNeish: *Theorems concerning transversals of the* $(n+1)$*-hedron in* n*-dimensional space.*

The line edges $A_i^0A_j^0 = a_{ij}$ of the $(n+1)$-hedron $A_1^0A_2^0\cdots A_{n+1}^0$ in n-space will be said to be divided antisymmetrically in the ratio $a_i:a_j$ by the point A_{ij}^0 if the ratio $A_i^0A_{ij}^0:A_{ij}^0A_j^0 = a_j:a_i$ where a_i, a_j, etc., are constants. In a triangle the 3 lines $A_i^0A_{jk}^0$ concur in the point A_{ijk}^0 which is said to divide the triangle antisymmetrically in the ratio $a_i:a_j:a_k$. The ratio $A_i^0A_{ijk}^0:A_{ijk}^0A_{jk} = a_j+a_k:a_i$ and $(A_i^0A_{jk}^0)^2 = (a_ja_{ik}^2 + a_ka_{ij}^2)/(a_j+a_k) - a_ja_ka_{jk}^2/(a_j+a_k)^2$. In a tetrahedron the following 7 lines concur in a point which is said to divide the tetrahedron antisymmetrically in a fixed ratio: (a) the 4 lines drawn from the vertices to the antisymmetric points of the opposite faces; (b) the 3 lines joining the antisymmetric points of the pairs of opposite edges. The three results (a) concurrence of transversals, (b) formulas for the lengths of the transversals, (c) the ratio of the segments of the transversals, stated above for 2 dimensions, are generalized to n dimensions.

7. Professor L. E. Dickson: *A new theory of the rational equivalence of linear transformations or pairs of bilinear forms.*

Let a linear transformation S on $\xi_1, \cdots, \xi_n$ with coefficients in any field F, the case of zero determinant not being excluded, replace any linear function x_1 of the ξ's by $x_1' = x_2$, x_2 by $x_2' = x_3, \cdots, x_{a-1}$ by $x_{a-1}' = x_a$, but x_a by a linear function $[x_1, \cdots, x_a]$ of those x's. Choose the leader x_1 so that a shall be the maximum length of all possible chains $x_1, \cdots, x_a$. If $n > a$, let b be the maximum length of a chain whose leader is linearly independent of $x_1, \cdots, x_a$. Continuing in this manner we may reduce S to a canonical form. The characteristic determinant of the partial transformation on any chain is divisible by that of the next chain. Apart from sign, these determinants are in reverse order the invariant factors, other than unity, of S. Two transformations S and T are similar in F (i. e., there exists a non-singular matrix B in F such that $BSB^{-1} = T$) if and only if the λ-matrices of S and T have the same invariant factors. Two pairs of n-rowed square matrices M, N and M_1, N_1 with elements in any field F, N and N_1 non-singular, are equivalent in F if and only if $M-\lambda N$ and $M_1-\lambda N_1$ have the same in-

variant factors. A like theorem follows when N and N_1 are singular, but the determinants of $\rho M+\sigma N$ and $\rho M_1+\sigma N_1$ are not zero identically in ρ and σ.

8. Professor Dickson: *Rational theory of pairs of bilinear forms in the singular case.*

The reduction of a pair of bilinear forms φ and ψ in the singular case in which the determinant of $u\varphi+v\psi$ is zero identically in u and v was first accomplished by Kronecker in the BERLINER SITZUNGSBERICHTE, 1890, pp. 1225–37. By a rational preliminary transformation he segregated component bilinear forms φ and ψ belonging to the non-singular case. For the latter, he employed the irrational canonical pair due to Weierstrass. Instead of these, we may employ the rational canonical pair which follows from the author's preceding paper. By purely rational transformation we may therefore reduce the initial φ and ψ to a canonical pair. The conclusion is that, in the singular case, two pairs of bilinear forms in the same $m+n$ variables with coefficients in any field F are equivalent in F if and only if they have the same invariant factors and the same minimal numbers, as defined by Kronecker.

9. Professor G. E. Wahlin: *On the solution of diophantine equations by means of ideals.*

This paper will appear in full in an early number of this BULLETIN.

10. Professor G. A. Miller: *Subgroup composed of the substitutions which omit a letter of a transitive group.*

The main object of this paper is to study transitive substitution groups G from the standpoint of the subgroups G_1 composed of all the substitutions which omit one letter. Among the theorems established are the following: The number of the transitive groups of degree $2k+1$ which have for their G_1 the group of order 2 and of degree $2k$ is the same as the number of the abstract abelian groups of order $2k+1$, k being an arbitrary positive integer. When G_1 is of order 2 and of degree 2^α the number of the possible degrees of the transitive groups which involve G_1 as the subgroup composed of all the substitutions which omit one letter is $\alpha+1$, where α is an arbitrary positive integer. A regular group of order

2^α, $\alpha > 2$, cannot appear as the G_1 of a transitive group of degree $2^\alpha + 2$. If a regular abelian group of order m is the G_1 of a transitive group of degree $m+k$ then $m \leq 2k$.

11. Professor W. E. Edington: *The transform in the abstract definitions of groups.*

In this paper a study is made of some of the conditions in special cases imposed by the transform $s_2^{-1}s_1s_2 = s_1^\alpha s_2^\beta$ in the definition of a group whose generators are s_1 and s_2. Also the general relations that must hold between the orders of the generators and the orders of the products of powers of the generators when $s_2^{-1}s_1s_2 = s_1^\alpha$ are determined and the definitions of certain infinite systems of groups are discussed. Some interesting properties of the transform considered as a substitution are also stated.

12. Professor H. T. Davis: *Asymptotic distribution of characteristic numbers in the problem of the elastic bar.*

The problem of the elastic bar may be generalized by generalizing the Sturmian boundary value problem of fourth order to which it leads. The author has previously derived sufficient conditions that the characteristic numbers of this Sturmian problem shall alternate with the λ-zeros of a solution of a certain Cauchy problem of fifth order. In this paper a study is made of the Cauchy problem and the asymptotic form of the λ-zeros determined in terms of the coefficients of the Sturmian system.

13. Professor H. Betz: *Surface transformations applied to special dynamical problems.*

This paper applies to actual dynamical problems with two degrees of freedom, the surface transformation methods developed by Poincaré and Birkhoff. It has for its particular object the discovery of periodic motions, and the examination of their stability. The "ring of Poincaré" is constructed graphically affording an insight into the nature of the totality of motions. One periodic motion is then singled out for detailed investigation and extensive calculations are made in order to determine its stability.

14. Professor E. L. Dodd: *The frequency law of a function of several variables with given frequency laws.*

The functions considered are characterized by such pro-

perties as continuity and limited variation; and formulas are proved for effecting the required integrations, with special attention to justifying changes in the order of integration. The general theory is applied to determine the frequency law for the sum of projections and for the sum of cubes, under stated hypotheses regarding the frequency law for the arguments.

15. Professor L. Ingold: *Note on isogonal trajectories of geodesics.*

In this note conditions are determined on the function $a(u, v)$ in order that the curves $a = \text{const.}$ shall be isogonal trajectories of a family of geodesics $c = \text{const.}$ This special case of "parallel displacement", so called, is easily treated by vector methods. The resulting formulas are quite similar to the formulas of the more general case.

16. Professor I. A. Barnett: *The parameter group of a continuous group in function space leaving a manifold invariant.*

This paper extends Kowalewski's work on groups in function space to groups involving any number of functions in any number of variables. Three applications of such groups are given, viz., the parameter group of a given group, the adjoint group, and the parameter group of a given group leaving manifolds in function space invariant. Functional invariants and covariants of the manifolds and the forms defining them are also considered.

17. Professor H. J. Ettlinger: *Existence theorems for differential equations.*

In this paper the author proves the existence of solutions of a first order ordinary differential equation by methods analogous to the proof already given for the integrability of continuous functions. Superior and inferior integrals are obtained, which are analogous to the Darboux upper and lower integrals.

18. Professor A. J. Kempner: *Polynomials and their residue systems.*

In a paper bearing the same title (TRANSACTIONS OF THIS SOCIETY, vol. 22) the residue system of a polynomial of a single variable with integral rational coefficients with respect

to a modulus m (m a rational integer, prime or composite) was examined. The investigation is now extended to the case of a polynomial of more than one variable.

19. Dr. Harold Hotelling: *Economic problems involving maxima of functionals.*

The assumption is made that an entrepreneur will do all in his power to maximize an integral representing his future profits, discounted according to remoteness in time. A second assumption—usually unjustified in a capitalist society—that a machine will be operated to full capacity, reduces the problem of depreciation of replaceable assets to one in ordinary differential calculus. This is elaborated, with a refinement obtained by solving a Volterra equation, in a paper offered to the JOURNAL OF THE AMERICAN STATISTICAL ASSOCIATION.

The first assumption alone reduces many problems of economics to those involved in maximizing functionals, and in particular to problems of the calculus of variations. The very puzzling problem of mining economics reduces by an immediate first integration of the Euler equations to a system of first-order equations. This solves the problem of depreciation of irreplaceable assets, both under competition and under monopoly. This application is in fact broader than economics, since eudaemonistic ethical theories hold that an integral of future happiness is to be maximized, and so reduce right conduct to a problem in functionals.

20. Professor J. B. Shaw: *On certain linear algebras.*

This paper is concerned with the synthetic construction of algebras from their sub-algebras; particularly those that satisfy a relation $\xi = \Phi(\xi)\sigma$, where ξ is to be any and every number of the sub-algebra, Φ is a linear operator, constant as to ξ, dependent upon σ, and σ is a number not in the sub-algebra. An instance is Dickson's representation of Cayley's octave algebra in the form $q_1 + q_2$, where q_1 and q_2 are quaternions, and $q = \bar{q}\omega$. To be associative, $\Phi(\xi_1)\cdot\Phi(\xi_2) = \Phi(\xi_1\xi_2)$, that is, Φ must leave the sub-algebra automorphic in multiplication. The octave algebra mentioned is not associative. Examples are given from abstract groups considered as algebras.

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