A NEW TYPE OF DOUBLE SEXTETTE CLOSED UNDER A BINARY (3,3) CORRESPONDENCE*

BY LOUISE D. CUMMINGS

1. *Introduction.* In connection with the Poncelet theorem that if a polygon of n sides can be inscribed in one conic and circumscribed to a second conic then an infinite number of these polygons exist, much investigation, both by elliptic functions and by algebraic methods, has been effected concerning the closure property for the (2,2) correspondence of the double binary forms. For the (3,3) correspondence the direct algebraic attack upon even the point sets of low orders has been, until rather recently, somewhat neglected.

Franz Meyer, among others, studied the (3,3) correspondence of four points and four planes and obtained the surprising result that if there is a first tetrahedron inscribed in one cubic curve and circumscribed to a second cubic curve there may not be a second tetrahedron, but if there is a second tetrahedron then an infinity of these tetrahedrons occur. The existence of one particular closed set of seven points and seven planes in a (3,3) correspondence, with the poristic property like the Poncelet polygons, was established in 1915 by White† and confirmed by Coble‡ who has investigated the general (m, n) correspondence. While Coble has not attempted an exhaustive classification, he has listed fourteen types, old and new, which are poristic configurations of double binary forms, among them one closed set of five points in a (3,3) correspondence. For periodic sets of six points in a (3,3) correspondence particular results are lacking. In this investigation of the (3,3) correspondence for $n = 6$, two non-congruent types of sextettes have been discovered by

---

* Presented to the Society, October 25, 1924.

† PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 1 (1915), p. 464.

‡ AMERICAN JOURNAL, vol. 43, No. 1 (Jan., 1921).

a simple algebraic method which is applicable to higher values of n.

A polyhedron of n faces, including the tetrahedron as the special case for $n = 4$, is suggested as a possible geometrical interpretation of the (3,3) correspondence for all values of n.

2. *The Special Problem.* The (3,3) correspondence

$$(a_0x^3 + b_0x^2 + c_0x + d_0)y^3 + (a_1x^3 + b_1x^2 + c_1x + d_1)y^2 +$$
$$(a_2x^3 + b_2x^2 + c_2x + d_2)y + (a_3x^3 + b_3x^2 + c_3x + d_3) = 0$$

contains fifteen constants. If the binaries x and y vary from 0 to ∞ and are independent projectively, this gives in general a non-closed correspondence. In an earlier paper* the author established the existence of one type of closed double sextette in a special (3,3) correspondence, possessing the poristic property like the Poncelet polygons in the (2,2) correspondence. This paper exhibits the nature of a second type of poristic sextette, and shows that only two non-congruent types exist. Two related gauche cubic curves are used for exemplification in order that this investigation may be compared readily with Meyer's study of the double quartette in which some of the generalized theorems appear to be erroneous. The points x lying on a twisted cubic K are represented by the six parameters a, b, c, d, e, f; the planes y osculating a second twisted cubic K' are represented by the parameters 1, 2, 3, 4, 5, 6 used as symbols; the double sextette is to be subject to the cyclic substitution $S \equiv (a\,b\,c\,d\,e\,f)(1\,2\,3\,4\,5\,6)$ and the correspondence is to be invariant under S and its powers. The relations between the points x and the planes y are shown in the two following mutual arrays I and I′, where the point $x = a$ lies in the triad of planes 1, 2, 4 or the plane $y = 1$ passes through the triad of points a, d, f.

* To appear in PROCEEDINGS INTERNATIONAL MATHEMATICAL CONGRESS, Toronto, 1924.

I

x	a	b	c	d	e	f
y	1	2	3	4	5	6
	2	3	4	5	6	1
	4	5	6	1	2	3

I′

y	1	2	3	4	5	6
x	a	b	c	d	e	f
	d	e	f	a	b	c
	f	a	b	c	d	e

3. *Determination of a Normal Form.* For the (3,3) correspondence with a closed cycle of six points, investigation shows that the form with fifteen constants may, without loss of generality, be replaced by a determinantal form which by the aid of the binary identities is reducible to an expression in four terms. Therefore the following trial form is set up initially,

$$F(x, y) = A\cdot xb\cdot xc\cdot xd\cdot y1\cdot y2\cdot y4 + B\cdot xc\cdot xd\cdot xa\cdot y2\cdot y3\cdot y5 + C\cdot xd\cdot xa\cdot xb\cdot y3\cdot y4\cdot y6 + D\cdot xa\cdot xb\cdot xc\cdot y4\cdot y5\cdot y1 = 0,$$

where for brevity xa represents the determinant of order 2 or the simple difference $x-a$.

This form visibly satisfies twelve of the conditions required by the array I, all on a, b, c, d; the application of the remaining six conditions furnished by the pairs $x = e$, $y = 5, 6, 2$ and $x = f$, $y = 6, 1, 3$ gives six equations linear in A, B, C, D such as

$$A\cdot eb\cdot ec\cdot ed\cdot 61\cdot 62\cdot 64 + B\cdot ec\cdot ed\cdot ea\cdot 62\cdot 63\cdot 65 + D\cdot ea\cdot eb\cdot ec\cdot 64\cdot 65\cdot 61 = 0.$$

Three of these equations determine the constants B/A, C/A, D/A, and with the binary identities allow the reduction of the form to any desired normal form such as the following:

$$\begin{aligned} F(x, y) = {} & 12\cdot 35\cdot 56\cdot ac\cdot cf\cdot xb\cdot xc\cdot xd\cdot y1\cdot y2\cdot y4 \\ & + 61\cdot 25\cdot 14\cdot bf\cdot ce\cdot xa\cdot xc\cdot xd\cdot y2\cdot y3\cdot y5 \\ & + 12\cdot 15\cdot 25\cdot ce\cdot cf\cdot xa\cdot xb\cdot xd\cdot y3\cdot y4\cdot y6 \\ & + 15\cdot 23\cdot 26\cdot cf\cdot de\cdot xa\cdot xb\cdot xc\cdot y4\cdot y5\cdot y1 = 0. \end{aligned}$$

The three remaining equations, which might conceivably impose redundant conditions upon the twelve parameters, give for the special case of the double sextette three independent relations E_1, E_2, E_3, equivalent to an invariant set of fifteen, where

$$E_1 \equiv \frac{ab\cdot cd}{ac\cdot bd} - \frac{23\cdot 45}{24\cdot 35} = 0; \quad E_2 \equiv \frac{ab\cdot ce}{ac\cdot be} - \frac{23\cdot 46}{24\cdot 36} = 0;$$

$$E_3 \equiv \frac{ab\cdot cf}{ac\cdot bf} - \frac{23\cdot 41}{24\cdot 31} = 0.$$

These fifteen invariant relations $E_1 = 0$, etc. express a unique projectivity of the x range to the y range, namely, that

$$\begin{matrix} a & b & c & d & e & f \\ \text{is projective to} & & & & & \\ 2 & 3 & 4 & 5 & 6 & 1. \end{matrix}$$

The form $F(x, y)$ thus admits by its construction one configuration of period 6; is this an isolated closed sextette, or does the form admit an infinitude of these configurations and so possess the poristic property?

4. *Conditions for the Porism of the Form.* To ascertain if the sextette is necessarily poristic, the twelve parameters which express the correspondence are replaced by the neighboring set $a + da$, $b + db$, $\cdots$, $6 + d6$, which must determine the same correspondence by the same formal equation. That is, the first differential must be a doubly cubic form identical with the original form $F(x, y)$, so that

$$F(x, y) + \frac{\partial F}{\partial a} da + \cdots + \frac{\partial F}{\partial 6} d6 \equiv M \cdot F(x, y).$$

A comparison of the coefficients of these two forms may be established by making use again of the eighteen pairs of points (x, y) given by the array I. These furnish eighteen equations of condition whose explicit form is derivable either by an algebraic process or by the following simple geometric method, in which for brevity da is replaced by a'. The array I$'$ shows that the form $F(x, y)$ regarded as a plane sextic is cut by the line $y = 1$ in the three points a, d, f and in a triple point at infinity, while the neighboring line $y = 1 + 1'$ cuts the curve in the points

$a+a'$, $d+d'$, $f+f'$ and in the same triple point at infinity. The slopes at the points $(a, 1)$, $(d, 1)$, $(f, 1)$ furnish the three equations

$$1' = \left(\frac{dy}{dx}\right)_{a,1} a' = \left(\frac{dy}{dx}\right)_{d,1} d' = \left(\frac{dy}{dx}\right)_{f,1} f'.$$

Replacing $\left(\frac{dy}{dx}\right)_{a,1}$ by $\left(\frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}\right)_{a,1}$ and making similar substitutions for the slopes at $(d, 1)$ and $(f, 1)$, we have the three equations

$$\left(\frac{\partial F}{\partial x}\right)_{a,1} a' + \left(\frac{\partial F}{\partial y}\right)_{a,1} 1' = 0; \quad \left(\frac{\partial F}{\partial x}\right)_{d,1} d' + \left(\frac{\partial F}{\partial 1}\right)_{d,1} 1' = 0;$$

$$\left(\frac{\partial F}{\partial x}\right)_{f,1} f' + \left(\frac{\partial F}{\partial y}\right)_{f,1} 1' = 0.$$

The substitution S applied to each of these three equations gives three cycles of six equations or eighteen equations of condition to be satisfied by the twelve differentials $a', b', \cdots, 5', 6'$. To establish the consistency of these eighteen equations and the uniqueness of the solution their number must be reduced to eleven. This can be done if these equations satisfy the two following identities.

$$(1)\quad \frac{1'}{2'}\cdot\frac{2'}{3'}\cdot\frac{3'}{1'} = 1; \qquad (2)\quad \frac{1'}{2'}\cdot\frac{2'}{3'}\cdot\frac{3'}{4'}\cdot\frac{4'}{5'}\cdot\frac{5'}{6'}\cdot\frac{6'}{1'} = 1,$$

where

$$\frac{1'}{2'} = \frac{13\cdot 16\cdot ad\cdot af\cdot bc\cdot ce}{23\cdot 26\cdot ae\cdot cf\cdot ab\cdot cd},$$

and the other ratios have values similar in form. Now both of these identities are found to be satisfied if

$$\frac{ac\cdot bd\cdot fe}{ae\cdot bc\cdot fd} = 1,$$

that is, if the three pairs of points f, c; a, d; b, e are in a quadric involution.

The identity (1) enables us, with the aid of the cyclic property, to discard six of the eighteen equations, while the identity (2) disposes of one more, leaving eleven linear equations for the determination of the eleven ratios

$$\frac{a'}{1'}, \frac{b'}{1'}, \ldots, \frac{6'}{1'}.$$

Therefore the eighteen equations become consistent and determine uniquely the ratios of the twelve differentials. This gives then the differential operator which generates in the Lie sense a singly infinite group of transformations of the curves K and K' into themselves, or, as we prefer to regard it, which transforms the sextette of points on the curve into a simple infinity of sextettes similarly related to the curves K and K'. Hence we have the following theorem.

THEOREM. *Six points selected arbitrarily on a twisted cubic K can form in a* (3,3) *correspondence an isolated closed system of the type here sought, but if the six points are specialized as three pairs in a quadric involution on the cubic then the sextette is poristic and slides along the cubic K while the six associated planes continue to osculate the same second curve K' of class* 3.

5. *The Secant-Axes of the Double Sextette.* In our closed set of six points and six planes the array I$'$ shows a line *ad* joining two points of K and lying in two planes 1 and 4 of K'—a line which is a secant of K and an axis of K'. So also two other secant-axes *be* and *cf* are obviously present. Now a well known theorem of Cremona fixes the number of secants of one twisted cubic, which are at the same times axes of another twisted cubic, at six, and while three of these secant-axes are in evidence the whereabouts of the remaining three requires further investigation.

6. *Meyer's Erroneous Statement concerning the Hurwitz Relation.* A study by this method of investigation of four

points and four planes in a (3,3) correspondence presents no difficulty and gives results in entire agreement with those obtained by Meyer for the double quartette. But a discrepancy arises from Meyer's further assertion that whenever the number of secant-axes is infinite, then the curves K and K' are necessarily in the Hurwitz relation, that is, these lines can be grouped in sets of six to form *the edges of infinitely many tetrahedrons,* all inscribed in K and circumscribed to K'. The Hurwitz relation exists for the double quartette but Meyer's assertion that it obtains in every poristic system is erroneous, as this double sextette shows. The secants through a are ad, ac, af and of these ad alone is a secant-axis. Now the Hurwitz relation requires at each point of the curve K three secant-axes, *whereas the sextette possesses one and no more*—other essential conditions also are not fulfilled.

This discrepancy in results can be harmonized only if for $F(x, y) = 0$, we find $A \equiv 0$, $B \equiv 0$, $C \equiv 0$, $D \equiv 0$, but as each of these coefficients is a single product of differences, this cannot occur unless points are assumed coincident.

7. *The Non-Congruent Types of Double Sextettes.* Since this correspondence is to be invariant under the cyclic substitution $S \equiv (abcdef)(123456)$ only three non-congruent types can occur, namely a with the triad 1, 3, 5; a with the triad 1, 2, 4; and a with the triad 1, 2, 3. The type a with 1, 3, 5 is not properly sextic and so is omitted; a with 1, 2, 4 is discussed in this paper. The double sextette in which a is associated with the planes 1, 2, 3 was investigated in an earlier paper* and was found to be poristic for six points specialized as elements in a quadric involution, precisely similar to the specialization required for the double sextette here discussed. In the type a; 1, 2, 3 the six secant-axes are all in evidence but the conditions essential for the Hurwitz relation are not fulfilled.

* Loc. cit.

8. *The Poristic Double Quintette.* This method of investigation has been applied to five points and five planes, and gives results in agreement with Coble's theorem concerning the existence of a double quintette in a (3,3) correspondence possessing the poristic property.

Under the substitution $S \equiv (a\,b\,c\,d\,e)(1\,2\,3\,4\,5)$ all possible types are congruent to the type a with the triad of planes 1, 2, 3. A normal form is

$$\begin{aligned} F(x, y) \equiv 45 \cdot ae \cdot xb \cdot xc \cdot xd \cdot y1 \cdot y2 \cdot y3 \\ + 51 \cdot be \cdot xc \cdot xd \cdot xa \cdot y2 \cdot y3 \cdot y4 \\ + 12 \cdot ce \cdot xd \cdot xa \cdot xb \cdot y3 \cdot y4 \cdot y5 \\ + 23 \cdot de \cdot xa \cdot xb \cdot xc \cdot y4 \cdot y5 \cdot y1 = 0. \end{aligned}$$

Conditions are imposed upon the ten differentials a', b', $\cdots$, $4'$, $5'$ by fifteen equations similar in form to those which occur for the double sextette. The fifteen equations are reducible to nine consistent equations, which determine uniquely the nine ratios

$$\frac{a'}{1'}, \frac{b'}{1'}, \cdots, \frac{5'}{1'},$$

if the set $a\ b\ c\ d\ e$ *is projective to* 2 3 4 5 1. No other conditions being imposed upon the parameters, a correspondence of period 5 can be determined with five arbitrary points for a fundamental set. The quintette is poristic; five secant-axes are ab, bc, cd, de, ea but the secant-axes do not fulfil the Hurwitz relation, so that here again Meyer's assertion is found incorrect.

9. *A Geometric Interpretation.* Since the Hurwitz relation does not hold for $n = 5$ nor for $n = 6$, there is for these cases no tetrahedron which slides along the curve K while its faces continue to osculate the second cubic K', as in the case of $n = 4$. However, one geometric figure applicable to all cases may be visualized in the following manner. From a polyhedron with n—1 lateral faces, such that no four of these lateral faces pass through a common point, cut a wedge whose upper and lower

bases intersect one of the lateral faces in the same straight line. The n planes which form the faces of the solid intersect in

$$\frac{n(n-1)(n-2)}{6}$$

vertices, and n of these vertices slide along the curve K while the n planes which form the faces continue to osculate the curve K'. This solid obviously becomes the tetrahedron for $n = 4$.

10. *Conclusion.* The method here employed is applicable to n points and n planes in a (3,3) correspondence. The determination of a normal form disposes of fifteen of the conditions imposed by the n-by-3 array I, leaving $3(n-5)$ conditions to be imposed upon the $2n$ parameters. The difficulty of eliminating from these $3(n-5)$ conditions those which are redundant increases rapidly with the increase in n, due to the increasing complexity in the coefficients of the normal form. The $3n$ equations arising in the comparison of $F(x, y)$ and its first derivative can be reduced to $2n-1$ equations, just sufficient for the determination of the $(2n-1)$ ratios $\frac{a'}{1'}, \frac{b'}{1'}, \cdots$, by two test identities similar to (1) and (2) used above. While the increase in n is accompanied always by an increase in the number of determinants in $\frac{a'}{1'}, \cdots$, nevertheless a further exploration of the field, by this method, does not present insurmountable difficulties for point sets of low orders.

The exhibition of these poristic point sets is one concrete step towards the realization of a suggestion offered by Mr. White in a paper on Poncelet polygons—"That all Poncelet systems are associated with linear involutions upon rational curves and that in this feature, possibly, lies even more promise of generalizations and discoveries than in Jacobi's brilliant and beautiful depiction by the aid of periodic functions".

VASSAR COLLEGE