# ON THE NUMBER OF REPRESENTATIONS OF AN INTEGER AS A SUM OR DIFFERENCE OF TWO CUBES*

### BY E. T. BELL

1. *Introduction and Summary.* Let $C(n)$ denote the number of integer solutions $(x, y) = (\xi, \eta)$, $\xi > 0$, $\eta > 0$, of $x^3 + y^3 = n$, the pair $(\xi, \eta)$, $(\eta, \xi)$ being considered as a single solution, and $D(n)$ the number of integer solutions $(x, y)$, $x > 0$, $y > 0$ of $x^3 - y^3 = n$, $n > 0$. If the pair $(\xi, \eta)$, $(\eta, \xi)$ in $C(n)$ be counted as two solutions, the total number is evidently $2C(n)$ or $2C(n) - 1$ according as $n$ is not or is the double of an integer cube $> 0$. No determination of $C(n)$, $D(n)$ seems to have been made. It will be of interest therefore to record forms of these functions *depending only upon the real divisors of* $n$, in analogy to the classical results for $x^2 \pm y^2 = n$. These forms also indicate fairly expeditious means for finding all the resolutions of $n$ into a sum or difference of two cubes.

We denote by $\psi(z)$ the well known function whose value is 1 or 0 according as $z$ is or is not an integer square $\geq 0$. In the sequel only integer arguments $z$ occur. For $S(n) \equiv C(n)$ or $D(n)$ we find the following:

(1)     $n \equiv 0 \bmod 3$,     $n \not\equiv 0 \bmod 9$,     $S(n) = 0$.

(2)     $n \equiv 0 \bmod 9$,     $S(n) = \sum \psi(4t - 3d^2)$,

the $\sum$ extending to all pairs $(t, d)$ of conjugate divisors of $n/9$ such that

$$\frac{1}{3}\sqrt[3]{n} < d \leq \frac{1}{3}\sqrt[3]{4n}, \text{ or } d < \frac{1}{3}\sqrt[3]{n},$$

according as $S = C$ or $D$.

(3)     $n \equiv \pm 1 \bmod 3$,     $S(n) = \sum \psi\left(\frac{4t - 3d^2}{3}\right)$,

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the $\sum$ extending to all pairs $(t, d)$ of conjugate divisors of $n$ such that $d \equiv \pm 1$ mod 3, and

$$\sqrt[3]{n} < d \leqq \sqrt[3]{4n}, \text{ or } d < \sqrt[3]{n},$$

according as $S = C$, $D$, the upper or the lower sign $\pm$ being taken throughout.

Corresponding to the exact formulas of Gauss and Eisenstein in the case of two squares for the number of lattice points in the circle of radius $\sqrt{n}$ with center at the origin, we have here

$$(4) \qquad \sum_{r=1}^{n} C(r) = \frac{1}{2} \left\{ [\sqrt[3]{n/2}] + \sum_{r=1}^{N} [\sqrt[3]{n-r^3}] \right\},$$

where $N = [\sqrt[3]{n}]$, and $[z] =$ the greatest integer $\leqq z$. This and the formulas for the circle are obvious cases of the following. Let $M(n)$ denote the total number of pairs of integers $(x, y)$, $x > 0$, $y > 0$, satisfying $n = ax^r + by^s$, where $a$, $b$, $r$, $s$ are integers $> 0$. Then

$$(5) \qquad \sum_{\alpha=1}^{n} M(\alpha) = \sum_{\alpha=1}^{N} [\sqrt[r]{(n-b\alpha^s)/a}], \quad N = [\sqrt[s]{n/b}].$$

2. *Proofs of* (1) — (5). Putting

$$n = dt = (x+y)(x^2 - xy + y^2) = x^3 + y^3,$$

we take $x + y = d$, $\quad x^2 - xy + y^2 = t$, and hence

$$3x^2 - 3dx + (d^2 - t) = 0.$$

The necessary and sufficient condition that this equation have an integer root is $4t - d^2 = 3w^2$, $w$ an integer. Since $4t - d^2$ and $d$ are of like parity, so also are $d$, $w$. Provided $d > w$, the values of $(x, y)$ are then $\frac{1}{2}(d \pm w)$ in either order, and hence from the definition of $C(n)$ we may take $w \geqq 0$, $x = \frac{1}{2}(d+w)$, $y = \frac{1}{2}(d-w)$. From $w^2 \geqq 0$, $d > w$, we have at once the first inequality stated in (3).

For a particular $d$ there is one and only one $t$, and hence not more than one $w$; so that each $d$ gives not more than one $(x, y)$, and the $(x, y)$'s for two different $d$'s are distinct.

The total number of pairs $(x, y)$ is therefore equal to the number of integer squares $(4t - d^2)/3$. Of the 9 possible cases only the following give $4t - d^2 \equiv 0 \mod 3$,

$$(n, d, t) \equiv (0, 0, 0), \quad (1, 1, 1), \quad (2, 2, 1) \mod 3.$$

In the case $(0, 0, 0)$ we set $(d, t) = (3d_1, 3t_1)$, and hence $n = 9n_1$ $(n_1 = d_1 t_1)$. Reduction of the conditions upon $d$, and final replacement of $(n_1, d_1)$ by $(n, d)$ with the proper qualifications gives $C(n)$ as stated in (2); the $C(n)$ of (1), (3) are equivalent to what precedes.

For $D(n)$ we proceed similarly from

$$n = dt = (x - y)(x^2 + xy + y^2) = x^3 - y^3,$$

setting $x - y = d$, and hence $3x^2 - 3dx + (d^2 - t) = 0$. The condition for an integer root is as before, and it is seen immediately that we can take $x = \frac{1}{2}(w + d)$, $y = \frac{1}{2}(w - d)$, $w > 0$, with the condition $w - d > 0$, which includes necessarily $w > 0$. Hence the sole condition is $d < \sqrt[3]{n}$.

The formula (5) is either evident or it is the immediate consequence of equating coefficients of $q^n$ in the expansions of

$$\frac{1}{1 - q} \times \{f(q^a) \ g(q^b)\}, \qquad \left\{\frac{f(q^a)}{1 - q}\right\} xg(q^b),$$

where

$$f(q) = \sum_0^\infty q^{n^r}, \qquad g(q) = \sum_0^\infty q^{n^s}, \qquad |q| < 1.$$

3. *Decomposition into Cubes.* The least favorable situation is that in which $n$ is so large that its resolution into factors is impracticable. It will be sufficient to indicate the process for $C(n)$ when $n \equiv 0 \mod 9$. We write down the $[\sqrt[3]{4n}/3] - [\sqrt[3]{n}/3]$ integers satisfying the first inequality in (2), test which of these divide $n/9$, obtaining thus the pairs $(d_i, t_i)$ $(i = 1, 2, \cdots)$ of conjugate divisors. If of these $i = a, b, \cdots$ give all pairs such that $4t_j - 3d_j^2$ $(j = a, b, \cdots)$ is a square, all the solutions of $x^3 + y^3 = n$ with $x > 0$, $y > 0$ are

$$x = \frac{1}{2}\left\{3d_j + \sqrt{4t_j - 3d_j^2}\right\}, \qquad y = \frac{1}{2}\left\{3d_j - \sqrt{4t_j - 3d_j^2}\right\}.$$

Numerous devices for shortening the computations are suggested by numerical work, whether or not the prime factor resolution of $n$ be feasible.

As an immediate consequence of (2) we note that 9 is the only prime multiple of 9 which is the sum of two cubes $> 0$; from (3) the only solution $x > 0$, $y > 0$ of $x^3 + y^3 = p^2$, $p$ prime, is $(x, y, p) = (1, 2, 3)$, etc. It is not difficult to obtain from (1) — (3) the known types of impossible equations $x^3 \pm y^3 = n$, except when $n$ is a cube, and some others that do not seem to have been stated.

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## CONTACT CURVES OF THE RATIONAL PLANE CUBIC*

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1. *Introduction.* Contact conics and hyperosculating curves of the rational cubic have been discussed by Winger.† Likewise some account has been given of curves of order $n$ which cut the cubic, rational or elliptic, in $(3n - 1)$ coincident points.‡ There remains the question of contact curves of order $n\,(n > 2)$ whose contacts are of lower orders. This paper considers that question for the rational cubic, with results which hold for $n \geqq 1$ and for contacts of any order.

If the cubic is taken in the canonical form

(1)        $x_1 = 3t^2, \qquad x_2 = 3t, \qquad x_3 = t^3 + 1,$

a necessary and sufficient condition that a set of $3n$

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