

NOTE ON THE PROJECTIVE GEOMETRY
OF PATHS

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1. *Projective Geometry of Paths.* It was first shown by Weyl† that the functions $\Gamma_{\alpha\beta}^i$ and the functions

$$(1) \quad \mathcal{A}_{\alpha\beta}^i = \Gamma_{\alpha\beta}^i + \delta_{\alpha}^i \psi_{\beta} + \delta_{\beta}^i \psi_{\alpha},$$

where ψ_{α} is an arbitrary covariant vector, and

$$\delta_{\alpha}^i = 0, \quad \text{for } i \neq \alpha; \quad = 1, \quad \text{for } i = \alpha,$$

define the same geometry of paths. This leads to the consideration of properties of the paths which are independent of the particular set of functions $\Gamma_{\alpha\beta}^i$ by means of which the paths are defined. Theorems expressing such properties constitute the projective geometry of paths. In the following note we give a few theorems belonging to the projective geometry of paths.

2. *Projective Tensors.* Theorems of the projective geometry of paths appear to have their statement in terms of what may be called *projective tensors*, i. e. tensors which are independent of the particular set of functions $\Gamma_{\alpha\beta}^i$ defining the paths. We shall show how a set of projective tensors may be derived by covariant differentiation from an n -uple of mutually independent vectors.

Let $h_{(\alpha)i}$ denote an n -uple of independent covariant vectors. Then the determinant

$$(2) \quad h = |h_{(\alpha)i}|$$

does not vanish identically. We may therefore define an n -uple of contravariant vectors $h^{(\alpha)i}$ as the cofactors of the

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† H. Weyl, GÖTTINGER NACHRICHTEN, 1921, p. 99. See also O. Veblen, PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 8 (1922), p. 347; and O. Veblen and T. Y. Thomas, TRANSACTIONS OF THIS SOCIETY, vol. 25, p. 557.

corresponding $h_{(\alpha)i}$ in the determinant h divided by h . Hence

$$(3) \quad h_{(\alpha)i} h^{(\alpha)j} = \delta_i^j,$$

and

$$(4) \quad h_{(\alpha)\alpha} h^{(\alpha)\alpha} = \delta_i^i,$$

where the left members represent a summation in α from $\alpha = 1$ to $\alpha = n$. It will be understood in the following that each index which appears twice in a term, once as a subscript and once as a superscript, is to be summed over the values 1 to n .

The change $[h_{(\alpha)i,j}]$ in the covariant derivative $h_{(\alpha)i,j}$ of the n -uple $h_{(\alpha)i}$, based on the functions $\Gamma_{\alpha\beta}^i$, when the Γ 's are replaced by the above functions \mathcal{A} , is

$$(5) \quad [h_{(\alpha)i,j}] = -h_{(\alpha)i} \psi_j - h_{(\alpha)j} \psi_i.$$

Hence

$$(6) \quad (n+1) \psi_i = -h^{(\alpha)\beta} [h_{(\alpha)\beta,i}].$$

The vector γ_i defined by

$$(7) \quad (n+1) \gamma_i = -h^{(\alpha)\beta} h_{(\alpha)\beta,i}$$

will therefore change by ψ_i when $\Gamma_{\alpha\beta}^i$ is replaced by $\mathcal{A}_{\alpha\beta}^i$, i.e.

$$(8) \quad [\gamma_i] = \psi_i.$$

Hence

$$(9) \quad h_{(\alpha)ij} = h_{(\alpha)i,j} + h_{(\alpha)i} \gamma_j + h_{(\alpha)j} \gamma_i$$

represents a set of n projective covariant tensors of the second order.

In a similar way we may treat the case of an n -uple of independent contravariant vectors $h^{(\alpha)i}$. This leads to a set of n projective mixed tensors of the second order given by

$$(10) \quad h_j^{(\alpha)i} = h^{(\alpha)i}_{,j} - h^{(\alpha)i} \lambda_j - \delta_j^i h^{(\alpha)\beta} \lambda_\beta$$

where the vector λ_i is defined as

$$(11) \quad (n+1) \lambda_i = h_{(\alpha)\beta} h^{(\alpha)\beta}_{,i},$$

in which the covariant n -uple $h_{(\alpha)i}$ is obtained from the

contravariant n -uple $h^{(\alpha)i}$ by dividing the cofactors of the elements of the determinant

$$h = |h^{(\alpha)i}|$$

by the determinant h . Other projective tensors will appear in the following paragraphs.

3. *The n -uple of Parallel Vectors.** If $h_{(\alpha)i}$ represents an n -uple of independent parallel covariant vectors for some set of functions $\Gamma_{\alpha\beta}^i$ defining the paths, then

$$(12) \quad h_{(\alpha)ij} = 0$$

identically. The condition (12) is also sufficient for the n -uple of independent covariant vectors $h_{(\alpha)i}$ to be parallel. Hence we have the following theorem.

THEOREM I. *A necessary and sufficient condition for the n -uple of independent covariant vectors $h_{(\alpha)i}$ to be parallel in the projective geometry of paths is that (12) be satisfied.*

Forming the equations

$$(13) \quad h_j^{(\alpha)i} = 0,$$

we may state the corresponding theorem for the case of a contravariant n -uple.

THEOREM II. *A necessary and sufficient condition for the n -uple of independent contravariant vectors $h^{(\alpha)i}$ to be parallel in the projective geometry of paths is that (13) be satisfied.*

4. *Reduction to the Euclidean Geometry.* The affine geometry of paths becomes a euclidean geometry if there exists an n -uple of independent parallel covariant vectors $h_{(\alpha)i}$. This is obviously a necessary condition and it is seen to be sufficient since the existence of an n -uple of parallel vectors $h_{(\alpha)i}$ leads to the equations

$$(14) \quad h_{(i)\sigma} B_{\alpha\beta\gamma}^{\sigma} = 0,$$

where $B_{\alpha\beta\gamma}^{\sigma}$ is the affine (ordinary) curvature tensor, and hence

* L. P. Eisenhart, PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 8 (1922), p. 207. See also O. Veblen and T. Y. Thomas, loc. cit., p. 589.

$$(15) \quad B_{\alpha\beta\gamma}^i = 0,$$

owing to the independence of the vectors $h_{(\alpha)i}$. Hence we have the following theorem.

THEOREM III. *A necessary and sufficient condition for the projective geometry of paths to be a euclidean geometry is that the Γ 's be such that there exists an n -uple of independent covariant vectors $h_{(\alpha)i}$ satisfying (12).*

A similar theorem may be stated for the n -uple of contravariant vectors.

5. *Reduction to the Riemann Geometry.* For the projective geometry of paths to be a Riemann geometry it is necessary and sufficient that the Γ 's be such that there exists a symmetric tensor $g_{\alpha\beta}$ whose determinant

$$(16) \quad g = |g_{\alpha\beta}|$$

does not vanish identically, and a vector ψ_i such that

$$(17) \quad \tilde{g}_{\alpha\beta,\gamma} = 0,$$

where $\tilde{g}_{\alpha\beta,\gamma}$ is the covariant derivative of $g_{\alpha\beta}$ based on the functions $\mathcal{A}_{\alpha\beta}^i$ given by (1). Writing (17) in the form

$$(18) \quad g_{\alpha\beta,\gamma} - 2g_{\alpha\beta}\psi_\gamma - g_{\gamma\beta}\psi_\alpha - g_{\alpha\gamma}\psi_\beta = 0,$$

where $g_{\alpha\beta,\gamma}$ is the covariant derivative of $g_{\alpha\beta}$ based on the functions $\Gamma_{\alpha\beta}^i$, and multiplying by the contravariant tensor $g^{\alpha\beta}$, formed in the ordinary manner from the tensor $g_{\alpha\beta}$, we have

$$(19) \quad 2(n+1)\psi_\gamma = g^{\alpha\beta}g_{\alpha\beta,\gamma}$$

Substituting this value of ψ_γ in the left member of (18) and denoting the resulting expression by $g_{\alpha\beta\gamma}$, we find

$$(20) \quad g_{\alpha\beta\gamma} = g_{\alpha\beta,\gamma} - \frac{g^{\mu\nu}}{n+1} \left(g_{\alpha\beta}g_{\mu\nu,\gamma} + \frac{1}{2}g_{\gamma\beta}g_{\mu\nu,\alpha} + \frac{1}{2}g_{\alpha\gamma}g_{\mu\nu,\beta} \right).$$

Hence (18) takes the form

$$(21) \quad g_{\alpha\beta\gamma} = 0.$$

The tensor $g_{\alpha\beta\gamma}$ defined by (20) may be shown to be projective.

The equations (21) constitute a necessary condition for the projective geometry of paths to be a Riemann geometry based on the tensor $g_{\alpha\beta}$ as the fundamental metric tensor. This condition is easily shown to be sufficient. Hence we have the following theorem.

THEOREM IV. *A necessary and sufficient condition for the projective geometry of paths to be a Riemann geometry is that the Γ 's be such that there exists a tensor $g_{\alpha\beta}$ which satisfies the equations (21).*

6. *Reduction to the Weyl Geometry.** Let us denote by $g_{\alpha\beta}$ a covariant symmetric tensor whose determinant g does not vanish identically, as in the preceding paragraph, and also by φ_α a covariant vector. Let us then form the equations

$$(22) \quad g_{\alpha\beta\gamma} = g_{\alpha\beta}\varphi_\gamma,$$

where $g_{\alpha\beta\gamma}$ is now defined by

$$(23) \quad g_{\alpha\beta\gamma} = g_{\alpha\beta,\gamma} - \frac{g^{\mu\nu}}{n+1} \left(g_{\alpha\beta}g_{\mu\nu,\gamma} + \frac{1}{2}g_{\gamma\beta}g_{\mu\nu,\alpha} + \frac{1}{2}g_{\alpha\gamma}g_{\mu\nu,\beta} \right) \\ + \frac{n}{n+1} \left(g_{\alpha\beta}\varphi_\gamma + \frac{1}{2}g_{\gamma\beta}\varphi_\alpha + \frac{1}{2}g_{\alpha\gamma}\varphi_\beta \right),$$

in which $g_{\alpha\beta,\gamma}$ is the covariant derivative of $g_{\alpha\beta}$ based on the functions $\Gamma_{\alpha\beta}^i$. The tensor $g_{\alpha\beta\gamma}$ defined by (23) is a projective tensor. We may prove the following theorem.

THEOREM V. *A necessary and sufficient condition for the projective geometry of paths to be a Weyl geometry is that the Γ 's be such that there exists a tensor $g_{\alpha\beta}$ and vector φ_α which satisfy the equations (22).*

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* By the Weyl geometry is meant the geometry used by Weyl as the basis of his combined theory of gravitation and electricity. See H. Weyl, *Raum, Zeit, Materie*, 4th ed., p. 113.