TWO GENERAL FUNCTIONAL EQUATIONS*

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The object of this paper is to discuss the functional equations

(1) $f(x+y)+g(x-y)=h(x)k(y), \quad h(x)\not\equiv 0, \quad k(x)\not\equiv 0,$

and

(2) $F(x+y)G(x-y)=H(x)+K(y), \quad F(x)\not\equiv 0, \quad G(x)\not\equiv 0,$

in which x and y are independent variables and $f(x)$, $g(x)$, $h(x)$, $k(x)$, $F(x)$, $G(x)$, $H(x)$, $K(x)$ are functions to be determined. Special cases of equations (1) and (2) have been discussed in the literature. Some of the more familiar special cases are

$$h(x)\equiv k(x)\equiv f(x)\equiv \psi(x), \quad g(x)\equiv 0,$$
$$g(x)\equiv k(x)\equiv f(x), \quad h(x)\equiv 2f(x),$$
$$G(x)\equiv 1, \ H(x)\equiv K(x)\equiv F(x)\equiv \varphi(x),$$
$$G(x)\equiv F(x), \ H(x)\equiv F^2(x), \ K(x)\equiv F^2(x)-1,$$
$$G(x)\equiv F(x), \ H(x)\equiv F^2(x), \ K(x)\equiv -F^2(x).$$

In this paper no relationships are assumed between the functions in equation (1) or the functions in equation (2). Furthermore, no restrictions (such as continuity, differentiability, etc.) are imposed on the functions. The variables, x and y, are not assumed real nor must they necessarily be complex. The author shows that the functions of equations (1) and (2) are expressible in terms of the functions $\varphi(x)$ and $\psi(x)$ which satisfy the Cauchy equations

(3) $$\varphi(x+y)=\varphi(x)+\varphi(y),$$

(4) $$\psi(x+y)=\psi(x)\psi(y),$$

given above as special cases of (1) and (2). The results of the paper are of sufficient generality to permit immediate

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application of information concerning solutions of (3) and (4) to equations (1) and (2).

If equation (1) *is satisfied, there are functions* $F(x)$, $G(x)$, $H(x)$, $K(x)$, *related to the functions* $f(x)$, $g(x)$, $h(x)$, $k(x)$, *which satisfy equation* (2).

To prove this replace x by $x+y$ and y by $x-y$ in equation (1). Then

$$h(x+y)k(x-y)=f(2x)+g(2y)$$

which is equation (2) if we define the functions of (2) by the equations

$$F(x)\equiv h(x),\quad G(x)\equiv k(x),\quad H(x)\equiv f(2x),\quad K(x)\equiv g(2x).$$

In like manner, *if equation* (2) *is satisfied, there are functions* $f(x)$, $g(x)$, $h(x)$, $k(x)$, *related to the functions* $F(x)$, $G(x)$, $H(x)$, $K(x)$, *which satisfy equation* (1). In view of these statements, it is sufficient if we discuss equation (1).

Since $k(x)\not\equiv 0$, there is some a such that $k(a)\neq 0$. We may replace y by $y+a$ in equation (1) with the result

$$f(x+y+a)+g(x-y-a)=h(x)k(y+a). \tag{5}$$

If we write

$$\begin{aligned} k(a)\bar{f}(x) &= f(x+a),\\ k(a)\bar{g}(x) &= g(x-a),\\ k(a)\bar{k}(x) &= k(x+a), \end{aligned}$$

equation (5) becomes at once of form (1) with the added condition that $\bar{k}(0)=1$. *We may therefore suppose, in what follows, that this transformation has been made and that* $k(0)=1$.

In equation (1), let $y=x$. Then

$$f(2x)=h(x)k(x)-g(0). \tag{6}$$

In like manner, if we let $y=-x$ in (1),

$$g(2x)=h(x)k(-x)-f(0). \tag{7}$$

Equations (6) and (7) show that $f(x)$ and $g(x)$ are determined, except for constants, by $h(x)$ and $k(x)$. In (1), replace x by $x+y$ and y by $x-y$. It follows that

$$h(x+y)k(x-y)=f(2x)+g(2y) \tag{8}$$
$$= h(x)k(x)+h(y)k(-y)-[f(0)+g(0)].$$

The auxiliary equation

$$h(2x)=[k(x)+k(-x)]h(x)-[f(0)+g(0)] \tag{9}$$

may be obtained by letting $y=x$ in (8). Interchanging x and y in (8), we have

$$h(x+y)k(y-x)=h(y)k(y)+h(x)k(-x)-[f(0)+g(0)]. \tag{10}$$

If, now, we add equations (8) and (10) and use equation (9), we find

$$h(x+y)[k(x-y)+k(y-x)]=h(2x)+h(2y), \tag{11}$$

which is transformed into

$$h(x+y)+h(x-y)=2h(x)E(y), \tag{12}$$

where $E(y)$ is the even component of $k(y)$, by replacing $2x$ by $x+y$ and $2y$ by $x-y$. The relation of equation (12) to equations (3) and (4) has been discussed by the writer.*

In order to find the odd component $S(x)$ of $k(x)$, we may subtract equation (10) from equation (8), whence

$$h(x+y)S(x-y)=h(x)S(x)-h(y)S(y). \tag{13}$$

In equation (13), replace x by $-x$ and y by $-y$ and add the equation thus found to (13). If we denote the odd component of $h(x)$ by $S_h(x)$, we have

$$S_h(x+y)S(x-y)=S_h(x)S(x)-S_h(y)S(y). \tag{14}$$

Replacing y by $-y$ in (14) leaves the right-hand member of the equation unchanged. Therefore

$$S_h(x+y)S(x-y)\equiv S_h(x-y)S(x+y), \tag{15}$$

whence, if $S_h(x)\not\equiv 0$,

$$S(x)\equiv \lambda S_h(x), \tag{16}$$

where λ is a constant.

* This BULLETIN, vol. 27 (1920), p. 302, equation (7).

If $S_h(x) \equiv 0$, $h(x)$ is an even function. Replacing x by $-x$ in (1), we have

$$f(-x+y)+g(-x-y) = h(x)k(y) = f(x+y)+g(x-y),$$

from which

$$g(x) = f(-x)+\mu,$$

where μ is a constant. Equation (1) may now be written in the form

$$(17) \qquad f(y+x)+f(y-x)+\mu = h(x)k(y).$$

Since $h(x) \not\equiv 0$, there is some value, say b, of x such that $h(b) \neq 0$. Replace x by b and y by $y+x$ in equation (17). The result is

$$(18) \qquad h(b)k(y+x) = f(y+x+b)+f(y+x-b)+\mu.$$

If we replace x by $-x$ in (18), we have

$$(19) \qquad h(b)k(y-x) = f(y-x+b)+f(y-x-b)+\mu$$

whence, from (18) and (19),

$$(20) \qquad h(b)[k(y+x)+k(y-x)] = [f(y+x+b)$$
$$+f(y-x-b)+\mu]+[f(y+x-b)+f(y-x+b)+\mu]$$
$$= h(x+b)k(y)+h(x-b)k(y)$$
$$= [h(x+b)+h(x-b)]k(y).$$

We may now define a function $C(x)$ by the equation

$$2h(b)C(x) = h(x+b)+h(x-b).$$

It is easily seen that $C(x)$ is an even function. Equation (20) may now be written in the form

$$(21) \qquad k(y+x)+k(y-x) = 2C(x)k(y).$$

If we interchange x and y in (21), we have

$$(22) \qquad k(x+y)+k(x-y) = 2k(x)C(y),$$

which is of the same form as equation (12). Since we are concerned, however, with the odd component of $k(x)$, we may change x to $-x$ and y to $-y$ in (22), and subtract the equation thus found from (22); whence we shall have

$$(23) \qquad S(x+y)+S(x-y) = 2S(x)C(y),$$

which is also of form (12), but which has the advantage of oddness (see reference under (12)).

The relationships of functions $f(x)$, $g(x)$, $h(x)$, $k(x)$ to the functions $\Phi(x)$ and $\psi(x)$ may now be summarized as follows (bearing in mind the transformation by which $k(0)=1$):

CASE A: $S_h(x)\not\equiv 0$. Either

$$E(x)=\frac{\psi(x)+\psi(-x)}{2},\qquad S_h(x)=\frac{\psi(x)-\psi(-x)}{2\alpha},$$

where α is a constant different from zero; or else

$$E(x)\equiv 1,\qquad S_h(x)=\Phi(x).$$

The even component of $h(x)$ is $\eta E(x)$, where η is a constant. In addition, we have

$$S(x)=\lambda S_h(x),$$

so that

$$h(x)=\eta E(x)+S_h(x),\qquad \bar{k}(x)=E(x)+S(x),$$

$$k(x)=k(a)\bar{k}(x-a),\qquad k(a)\neq 0,$$

$$f(2x)=h(x)k(x)-g(0),\qquad g(2x)=h(x)k(-x)-f(0).$$

CASE B: $S_h(x)\equiv 0$. In this case we have

$$E(x)=\frac{\psi_1(x)+\psi_1(-x)}{2},$$

and either

$$S(x)=\frac{\psi_2(x)-C(x)}{\alpha},$$

where α is a constant different from zero; or else

$$S(x)=\Phi(x)\text{ and }C(x)\equiv 1,$$

where each of the functions $\psi_1(x)$ and $\psi_2(x)$ satisfies (4) and

$$2h(b)C(x)=h(x+b)+h(x-b),\ h(b)\neq 0.$$

In this case,

$$h(x)=\eta E(x),$$

where η is a constant; and

$$\bar{k}(x)=E(x)+S(x),\qquad k(x)=\bar{k}(a)k(x-a),\qquad k(a)\neq 0,$$

$$f(2x)=h(x)k(x)-g(0),\qquad g(2x)=h(x)k(-x)-f(0).$$

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