# FUNCTIONAL INVARIANTS, WITH A CONTINUITY OF ORDER $p$, OF ONE-PARAMETER FREDHOLM AND VOLTERRA TRANSFORMATION GROUPS*

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1. *Object of the Paper.* In the present paper sufficient conditions are given for the invariance of functionals $f[y(\tau_0'), y'(\tau_0'), y''(\tau_0'), \cdots, y^{(p)}(\tau_0')]$ with a Volterra variation under given arbitrary Fredholm groups

$$(1) \qquad \delta y(x) = \left[\int_0^1 H(x,s) y(s)\, ds\right] \delta a$$

with a kernel $H(x,s)$ of the type†

$$(2) \qquad H(x,s) = \eta(x)\psi(s).$$

These sufficient conditions are given in the form of func-

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* Presented to the Society, October 7, 1924.

† In this BULLETIN, vol. 30 (July, 1924), pp. 338–344, the writer considered analytic functionals $f[y(\tau_0'), y'(\tau_0')]$ admitting Fredholm groups (1). In the process of finding sufficient conditions for invariance, a certain amount of specialization on the kernel $H(x,s)$ was necessary. The specialization however was erroneously made more than necessary due to an error in formula (16). The reader will verify the statement that the corrected formula (16) becomes

$$f_{i,k}(t_1, \cdots, t_i; t_{i+1}, \cdots, t_{i+k-1}, t)$$
$$= (-1)^k \varphi(t)\varphi(t_{i+k-1})\varphi(t_{i+k-2})\cdots\varphi(t_{i+1}) f_{i+k,0}(t_{i+1}, t_{i+2}, \cdots, t_{i+k-1}, t, t_1, \cdots, t_{i,j}).$$

With the aid of this corrected formula, we can show readily that the kernel $H(x,s)$ will have to be of the form $\eta(x)\psi(s)$ and hence $\varphi(t) = \eta(t) \Big/ \frac{d\eta(t)}{dt}$. Consequently the $f_{i,k}$'s need not be symmetric in all their arguments. Theorem II of the cited paper becomes true when the above changes are introduced.

In the cited paper we also need to make the following correction: substitute "It is sufficient that" for the wording "We may now apply Lemma 2 of I. D. I. V.; doing so we find" immediately preceding equation (12).

tional equations with partial functional derivatives. Of special interest are the linear functionals in which case necessary as well as sufficient conditions for invariance are given. We further demonstrate a theorem which shows the unique rôle played by a linear functional of $y(\tau)$ and $y'(\tau)$ that admits a given arbitrary *Volterra* group of transformations

$$(3) \qquad \delta y(x) = \left[\int_0^x H(x,s)y(s)\,ds\right]\delta a$$

with

$$H(s,s) \neq 0.$$

2. *A Sufficient Condition for Invariance.* We proceed to prove the following theorem.

THEOREM I. *A sufficient condition that a functional* $f[y(\tau_0'), y'(\tau_0'), \cdots, y^{(p)}(\tau_0')]$ *with a Volterra variation and continuous functional derivatives admit a given arbitrary Fredholm group of transformations* (1) *with a kernel* $H(x,s)$ *of type* (2) *(*$\eta(x)$ *being assumed* to be such that all the* $\eta^{(i)}(x)/\eta(x)$*'s are continuous in the interval* 0,1) *is that it satisfy the completely integrable functional equation with partial functional derivatives*

$$(4) \quad f_y(t) = -\frac{\eta'(t)}{\eta(t)}f_{y'}(t) - \frac{\eta''(t)}{\eta(t)}f_{y''}(t) - \cdots - \frac{\eta^{(p)}(t)}{\eta(t)}f_{y^{(p)}}(t),$$

*where* $f_{y^{(i)}}(t)$ *represents the partial functional derivative of* $f[y, y', y'', \cdots, y^{(p)}]$ *with respect to* $y^{(i)}(\tau)$ *taken at the point* $t$, *and where*

$$\eta^{(i)}(t) \equiv \frac{d^i\eta(t)}{dt^i}.$$

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* To shorten the statements of the theorems that follow in this paper we shall always assume this restriction on the $\eta(x)$ found in the Fredholm kernel (2). Moreover it is scarcely necessary to state here that we assume that the derivatives of $\eta(x)$ up to and including the $p$th derivative exist or else our functional equations will have no meaning.

PROOF. A necessary and sufficient condition that $f[y, y', \cdots, y^{(p)}]$ admit a given arbitrary Fredholm group (1) with a kernel $\eta(x)\psi(s)$ is that

$$\delta f[y, y', \cdots, y^{(p)}] \equiv \delta a \int_0^1 \Big[ f_y(t)\eta(t)\int_0^1 \psi(s)y(s)\,ds + f_{y'}(t)\eta'(t)\int_0^1 \psi(s)y(s)\,ds + + \cdots + f_{y^{(p)}}(t)\eta^{(p)}(t)\int_0^1 \psi(s)y(s)\,ds\Big]dt \equiv 0.$$

Hence it is sufficient that

$$\eta(t)f_y(t) + \eta'(t)f_{y'}(t) + \eta''(t)f_{y''}(t) + \cdots + \eta^{(p)}(t)f_{y^{(p)}}(t) = 0,$$

which can be written in the form (4).

Paul Lévy* has treated extensively the integrability conditions and the Cauchy problem for functional equations with partial functional derivatives in the case of functionals of two independent functions. Cauchy problems and integrability conditions for equations of type (4) involving functionals of $p$ functions can be considered by an extension of Lévy's work.

We now consider the functional equation

$$(5) \quad f_{y_p}(t) = g_1(t)f_{y_1}(t) + g_2(t)f_{y_2}(t) + \cdots + g_{p-1}(t)f_{y_{p-1}}(t).$$

Assume that each of the $p$ functional arguments $y_1(\tau), y_2(\tau), \cdots, y_p(\tau)$ is expressed in terms of two parameters $\lambda$ and $\mu$. Let $\delta_\lambda$ be the variation operator when $\lambda$ changes and let $\delta_\mu$ be the variation operator when $\mu$ changes. By an obvious extension of Lévy's work, it can be shown that a necessary and sufficient condition that the functional equation (5) be integrable† is that

* Cf. his book *Leçons d'Analyse Fonctionnelle*, Part II.

† Cf. Lévy's book, § 78, Part I; note at the end of Chapter I, Part II; §§ 45, Part II. Lévy assumes the existence of a solution in getting his integrability conditions. In our case, however, it is possible to get explicit solutions by a direct computation. See § 4 of this paper.

(6) $$\delta_\mu \delta_\lambda f[y_1, y_2, \cdots, y_p] \equiv \delta_\lambda \delta_\mu f[y_1, y_2, \cdots, y_p].$$

A necessary and sufficient condition* that the integrability condition (6) hold is that $E_{qk}$ be adjoint† to $E_{kq}(k, q = 1, 2, \cdots, p)$, where $E_{qk}$ stands for the linear functional of $\delta y_k$ found in the expression for $\delta f_{y_q}(t)$.

If for $y_p(\tau) \equiv y_{0p}(\tau), f[y_1, y_2, \cdots, y_p]$ is a given arbitrary functional of $y_1, y_2, \cdots, y_{p-1}$, then the $f_{y_i}(t)$'s $(i = 1, 2, \cdots, p-1)$ are known for $y_p(\tau) \equiv y_{0p}(\tau)$ and hence the linear functionals $E_{kq}(k, q = 1, 2, \cdots, p-1)$ are known and are furthermore respectively adjoint to the $E_{qk}$'s. To compute the $E_{pi}$'s and the $E_{ip}$'s $(i = 1, 2, \cdots, p)$, we take the variation of both sides of (5). Doing so, we get

(7) $$\delta f_{y_p}(t) = g_1(t)\delta f_{y_1}(t) + g_2(t)\delta f_{y_2}(t) + \cdots + g_{p-1}(t)\delta f_{y_{p-1}}(t),$$

an identity in $\delta y_1, \delta y_2, \cdots, \delta y_p$.

Consequently on equating terms in $\delta y_i$, we get

(8) $$\begin{aligned} E_{pi}[\delta y_i(\tau)/t] &= g_1(t)E_{1i}[\delta y_i(\tau)/t] + g_2(t)E_{2i}[\delta y_i(\tau)/t] + \\ &\cdots + g_{p-1}(t)E_{p-1,i}[\delta y_i(\tau)/t], \qquad (i = 1, 2, \cdots, p). \end{aligned}$$

From (8) we see that the $E_{pi}$'s $(i = 1, 2, \cdots, p-1)$ are given in terms of known linear functionals. Hence, imposing the condition that $E_{pi}$ be adjoint to $E_{ip}(i = 1, 2, \cdots, p-1)$, we get from (8)

(9) $$\begin{aligned} E_{ip}[\delta y_i(\tau)/t] &= E_{i1}[g_1 \delta y_i/t] + E_{i2}[g_2 \delta y_i/t] + \\ &\cdots + E_{i,p-1}[g_{p-1}\delta y_i/t], \quad (i = 1, 2, \cdots, p-1). \end{aligned}$$

Finally in order that the integrability conditions (6) be surely satisfied we must have $E_{pp}[\delta y_p(\tau)/t]$ self-adjoint. On substituting the expression for $E_{ip}$, as given in (9), in the expression for $E_{pp}$ in (8), we can see with a little computation that $E_{pp}$ is automatically self-adjoint. Hence the functional equation (5) is completely integrable. Since $p$ can be any positive integer, it follows that (4) is also a completely integrable equation. This completes the proof of our theorem.

* This condition can readily be found by computation.

† For a definition of adjointness see Lévy's book, § 73, Part I.

The following corollaries are immediate.

COROLLARY 1. *There is one and only one solution* $f[y, y', y'', \cdots, y^{(p)}]$ *of* (4), *regarded momentarily as a functional of* $p+1$ *independent arguments, such that for a given initial functional value of the first argument*

$$y(\tau) \equiv y_0(\tau),$$

$f[y, y', y'', \cdots, y^{(p)}]$ *has arbitrary values as a functional of its last* $p$ *arguments* $y'(\tau), y''(\tau), \cdots, y^{(p)}(\tau)$.

COROLLARY 2. *There always exist functionals* $f[y(\tau_0'), y'(\tau_0'), \cdots, y^{(p)}(\tau_0')]$, *with a Volterra variation, admitting a given arbitrary group of Fredholm transformations* (1) *with a kernel of type* (2).

3. *Linear Functionals Continuous of Order* $p$; *Necessary and Sufficient Conditions.** Of special interest are the linear functionals with a Volterra variation; i. e., functionals of the form

$$f_0 + \int_0^1 f_1(t) y(t)\,dt + \int_0^1 f_2(t) y'(t)\,dt + \cdots + \int_0^1 f_{p+1}(t) y^{(p)}(t)\,dt, \tag{10}$$

where $p$ denotes any positive integer or zero; and all the $f_i(t)$'s are assumed continuous functions of $t$ in the interval (0, 1). We shall demonstrate the following theorem.

THEOREM II. *A necessary and sufficient condition that a linear functional* (10) *of continuity order* $p$ *admit a given arbitrary group of Fredholm transformations* (1) *with a kernel of type* (2)† *is that it satisfy the completely integrable functional equation*

$$f_y(t) = -\frac{\eta'(t)}{\eta(t)} f_{y'}(t) - \frac{\eta''(t)}{\eta(t)} f_{y''}(t) - \cdots - \frac{\eta^{(p)}(t)}{\eta(t)} f_{y^{(p)}}(t) + \frac{\omega(t)}{\eta(t)}, \tag{11}$$

* Cf. G. C. Evans, *The Cambridge Colloquium*, 1916, Lecture III, § 3.

† We further restrict ourselves in this theorem to kernels of type (2) with $\eta(x) \neq 0$ in the interval (0, 1).

*where* $\omega(t)$ *is a given arbitrary continuous function satisfying the equation*

$$\int_0^1 \omega(t)dt = 0. \tag{12}$$

Proof. Evidently a necessary and sufficient condition for invariance is that

$$\left[\int_0^1 y(s)\psi(s)ds\right]\left[\int_0^1 \{f_y(t)\eta(t) + f_{y'}(t)\eta'(t) + \cdots + f_{y^{(p)}}(t)\eta^{(p)}(t)\}\,dt\right] \equiv 0$$

in $y$. Hence it follows, since $\psi(s) \not\equiv 0$* and since all the functional derivatives of $f[y, y', y'', \cdots, y^{(p)}]$ are point functions, that

$$\int_0^1 \{f_y(t)\eta(t) + f_{y'}(t)\eta'(t) + \cdots + f_{y^{(p)}}(t)\eta^{(p)}(t)\}\,dt = 0,$$

a condition which evidently can be expressed in the form of the functional equation (11)†.

The following important theorem is immediate.

Theorem III. *Given any continuous function* $\omega(t)$ *such that* (12) *holds, and given the initial conditions in the Cauchy problem for equation* (11), *then there is one and only one linear functional* (10) *of continuity order* $p$ *admitting a given arbitrary Fredholm group of transformations* (1) *with a kernel of type* (2) *and taking on the given initial conditions.*

We remark here that since $\psi(s)$ does not enter into the final conditions for invariance for the linear as well as for the non-linear functional, there is an infinitude of Fredholm groups (1) with a kernel of type (2) leaving one functional, of the types considered, invariant.

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* For if $\psi(s) \equiv 0$, then our transformation degenerates into the identical one and hence there is no problem.

† Equation (11) can readily be shown to be completely integrable by a slight modification of the reasoning involved in treating equation (4).

4. *Explicit Expressions for the Functional Invariants.* To get explicit expressions for the functional invariants, we can assume $f[y, y', y'', \cdots, y^{(p)}]$ to be expansible in a series of functionals. For instance we can assume $f[y, y', y'', \cdots, y^{(p)}]$ to be an analytic functional of its $p+1$ functional arguments $y(\tau), y'(\tau), y''(\tau), \cdots, y^{(p)}(\tau)$ and then by a reasoning similar to that found in one* of the writer's papers, recurrence formulas can be computed yielding the desired functional invariants in terms of the given initial conditions. For the sake of brevity, we here give only a simple illustration.

Consider the problem of finding the functional $f[y(\tau_0'), y'(\tau_0'), y''(\tau_0')]$ of the form

$$
\begin{aligned}
&\int_0^1 f_{100}(t)y(t)\,dt + \int_0^1 f_{010}(t)y'(t)\,dt + \int_0^1 f_{001}(t)y''(t)\,dt \\
&\qquad + \frac{1}{2!}\Big[\int_0^1\int_0^1 f_{200}(t_1,t_2)y(t_1)y(t_2)\,dt_1\,dt_2 \\
&\qquad + \int_0^1\int_0^1 f_{020}(t_1,t_2)y'(t_1)y'(t_2)\,dt_1\,dt_2 \\
&\qquad + \int_0^1\int_0^1 f_{002}(t_1,t_2)y''(t_1)y''(t_2)\,dt_1\,dt_2 \qquad (13)\\
&\qquad + 2\int_0^1\int_0^1 f_{110}(t_1,t_2)y(t_1)y'(t_2)\,dt_1\,dt_2 \\
&\qquad + 2\int_0^1\int_0^1 f_{101}(t_1,t_2)y(t_1)y''(t_2)\,dt_1\,dt_2 \\
&\qquad + 2\int_0^1\int_0^1 f_{011}(t_1,t_2)y'(t_1)y''(t_2)\,dt_1\,dt_2\Big]
\end{aligned}
$$

admitting a given arbitrary Fredholm group of transformations (1) with a kernel of type (2).

We can, without any loss of generality, assume $f_{200}(t_1, t_2)$, $f_{020}(t_1, t_2)$ and $f_{002}(t_1, t_2)$ to be symmetric in $t_1$ and $t_2$. We further assume that all the $f_{ikl}$'s are continuous functions of all their arguments.

* *Integro-differential expressions invariant under Volterra's group of transformations,* ANNALS OF MATHEMATICS, vol. 26, p. 181 (March, 1925).

From our previous theory we see that for invariance it is sufficient that $f[y(\tau_0'), y'(\tau_0'), y''(\tau_0')]$ of type (13) satisfy the completely integrable functional equation

$$(14) \qquad f_y(t) = -\frac{\eta'(t)}{\eta(t)} f_{y'}(t) - \frac{\eta''(t)}{\eta(t)} f_{y''}(t).$$

Assume the initial conditions for this equation to be given for

$$y(\tau) \equiv 0$$

in the form

$$(15) \quad \begin{aligned} f[0, y'(\tau_0'), y''(\tau_0')] \equiv & \int_0^1 f_{010}(t) y'(t)\,dt + \int_0^1 f_{001}(t) y''(t)\,dt \\ & + \frac{1}{2!}\left[\int_0^1\int_0^1 f_{020}(t_1, t_2) y'(t_1) y'(t_2)\,dt_1\,dt_2 \right. \\ & + \int_0^1\int_0^1 f_{002}(t_1, t_2) y''(t_1) y''(t_2)\,dt_1\,dt_2 \\ & \left. + 2\int_0^1\int_0^1 f_{011}(t_1, t_2) y'(t_1) y''(t_2)\,dt_1\,dt_2\right]. \end{aligned}$$

Calculating the partial functional derivatives of $f[y(\tau_0'), y'(\tau_0'), y''(\tau_0')]$ and substituting them in the functional equation (14) we get, on equating coefficients of similar terms in $y, y'$, and $y''$, the unique determination of the four unknown $f_{ikl}$'s in terms of the five initially known $f_{ikl}$'s. Thus we get after computation

$$\begin{aligned} f_{100}(t) &= -\frac{\eta'(t)}{\eta(t)} f_{010}(t) - \frac{\eta''(t)}{\eta(t)} f_{001}(t), \\ f_{110}(t_1, t_2) &= -\frac{\eta'(t_1)}{\eta(t_1)} f_{020}(t_1, t_2) - \frac{\eta''(t_1)}{\eta(t_1)} f_{011}(t_2, t_1), \\ f_{101}(t_1, t_2) &= -\frac{\eta'(t_1)}{\eta(t_1)} f_{011}(t_1, t_2) - \frac{\eta''(t_1)}{\eta(t_1)} f_{002}(t_1, t_2), \\ f_{200}(t_1, t_2) &= \frac{\eta'(t_1)}{\eta(t_1)}\frac{\eta'(t_2)}{\eta(t_2)} f_{020}(t_2, t_1) \\ & \quad + \frac{\eta'(t_1)}{\eta(t_1)}\frac{\eta''(t_2)}{\eta(t_2)} f_{011}(t_1, t_2) + \frac{\eta'(t_2)}{\eta(t_2)}\frac{\eta''(t_1)}{\eta(t_1)} f_{011}(t_2, t_1) \\ & \quad + \frac{\eta''(t_1)}{\eta(t_1)}\frac{\eta''(t_2)}{\eta(t_2)} f_{002}(t_2, t_1). \end{aligned}$$

5. *Functional Invariants of Volterra Groups.* It is of interest at this point to give certain results as regards linear functionals of continuity order $p$ of type (10) which admit a given arbitrary one-parameter Volterra group of infinitesimal transformations.

The author has treated extensively in another paper* the problem of finding functionals $f[y(\tau_0'), y'(\tau_0')]$ which admit a given arbitrary Volterra group of transformations defined by (3).

In particular it was shown that a necessary and sufficient condition that a linear functional $f[y(\tau_0'), y'(\tau_0')]$ of the type

$$f_0 + \int_0^1 f_1(t) y(t)\,dt + \int_0^1 f_2(t) y'(t)\,dt \tag{16}$$

($f_1(t)$ and $f_2(t)$ assumed continuous in $(0,1)$) admit a given arbitrary Volterra group of transformations (3) with $H(s,s) \neq 0$ is that it satisfy the completely integrable functional equation

$$f_{y'}(s) = \int_0^1 L(s,r) f_y(r), \tag{17}$$

where

$$\begin{aligned} L(s,r) &= \int_s^r k(s,r') \frac{H(r,r')}{H(r',r')}\,dr' - \frac{H(r,s)}{H(s,s)}, & r \geqq s, \\ &= 0, & r < s, \end{aligned}$$

and where $k(s,r')$ is the reciprocal kernel of

$$-\frac{1}{H(s,s)} \frac{\partial H(r',s)}{\partial r'}.$$

It was also shown that there are no linear functionals

$$f_0 + \int_0^1 f(t) y(t)\,dt$$

($f(t)$ assumed continuous in $(0,1)$) other than constants, which admit a given arbitrary Volterra group of transformations (3) with $H(s,s) \neq 0$.

With these facts in mind we proceed to prove the following theorem.

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* *Integro-differential expressions,* loc. cit.

THEOREM IV. *The linear functionals* (16) *are the only linear functionals* (10) *other than constants that can admit a given arbitrary Volterra group of transformations** (3) *with* $H(s, s) \neq 0$.

The proof of this theorem can be made to depend on the following lemma.

LEMMA. *Let* $F_0(x), F_1(x), \cdots, F_k(x)$ *be continuous functions in the interval* $(0, 1)$. *If*

$$(18) \quad \int_0^1 [F_0(x)y(x) + F_1(x)y'(x) + \cdots + F_k(x)y^{(k)}(x)]\,dx = 0$$

*for all possible forms of the function* $y(x)$, *having a continuous kth derivative* $y^{(k)}(x)$ *in* $(0, 1)$, *then it is necessary*† *that* $F_k'(x)$ *exist and be continuous in* $(0, 1)$ *and that* $F_k(0) = F_k(1) = 0$.

PROOF. Assume

$$(19) \quad \begin{aligned} y(0) &= y(1) = y'(0) = y'(1) = \\ &\cdots = y^{(k-1)}(0) = y^{(k-1)}(1) = 0. \end{aligned}$$

Then making use of a well known formula‡ and by an evident use of Dirichlet's formula, we get

$$(20) \quad \begin{aligned} &\int_0^1 y^{(k)}(t)\Bigg[F_k(t) + \int_t^1 \Big\{F_{k-1}(x) + F_{k-2}(x)(x-t) + \\ &\cdots + F_1(x)\frac{(x-t)^{k-2}}{(k-2)!} + F_0(x)\frac{(x-t)^{k-1}}{(k-1)!}\Big\}\,dx\Bigg]\,dt = 0 \end{aligned}$$

for all $y$ of our hypotheses satisfying (19). It follows§ therefore that

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* We note, without any further statement, that whenever we shall need the $p$th extended Volterra group of transformations, we shall assume that all the derivatives of $H(x, s)$ and $H(s, s)$ that are found in this extended transformation exist and are continuous.

† The reader can, without difficulty, push the argument until necessary and sufficient conditions on all the $F_i(t)$'s are found by successive applications of our lemma and by successive valid integrations by parts; but the lemma as stated is sufficient for our purpose.

‡ Cf. Hadamard, *Calcul des Variations*, vol. I, p. 201.

§ Cf. Hadamard, loc. cit.

$$
\begin{aligned}
F_k(t) = -\int_t^1 \Big\{ & F_{k-1}(x) + F_{k-2}(x)(x-t) + \\
(21) \qquad & \cdots + F_1(x)\frac{(x-t)^{k-2}}{(k-2)!} + F_0(x)\frac{(x-t)^{k-1}}{(k-1)!}\Big\}dx \\
& + l_0 + l_1(1-t) + l_2(1-t)^2 + \cdots + l_{k-1}(1-t)^{k-1},
\end{aligned}
$$

where the $l_i$'s are arbitrary constants. The derivative of the right hand side of (21) exists and is continuous in $(0,1)$ and hence the derivative $F_k'(t)$ of $F_k(t)$ exists and is continuous in $(0,1)$. Dropping now the restrictions (19) and calculating $F_k'(t)$ from (21) we proceed as follows. We substitute the expression for $F_k(t)$ in (18) and perform the now valid integration by parts. Carrying out $k-1$ valid successive integrations by parts starting with the term involving $y^{(k-1)}(t)$ and ending with the resultant term involving $y'(t)$, we get, on putting

$$
(22) \qquad \begin{aligned} y(0) = y(1) = y'(0) = y'(1) = \\ \cdots = y^{(k-2)}(0) = y^{(k-2)}(1) = 0, \end{aligned}
$$

$$
(23) \qquad [F_k(t)y^{(k-1)}(t)]_0^1 = 0
$$

for all $y$'s satisfying (22). In order that (23) may hold it is necessary and sufficient that $F_k(0) = F_k(1) = 0$. This completes the proof of our lemma.

A necessary and sufficient condition that a linear functional (10) admit a given arbitrary Volterra group of transformations (3) is that

$$
\begin{aligned}
\int_0^1 [ & y(t)h_1(t) + y'(t)h_2(t) + \\
& \cdots + y^{(p-2)}(t)h_{p-1}(t) + y^{(p-1)}(t)H(t,t)f_{p+1}(t)]\,dt \equiv 0
\end{aligned}
$$

in $y$, where $h_i(t)$ $(i = 1, 2, \cdots, p-1)$ are easily calculated expressions whose explicit form is not necessary to the argument. On applying the lemma just proved we conclude that it is necessary that the derivative of $H(t,t)f_{p+1}(t)$ exist and be continuous and that

$$
H(1,1)f_{p+1}(1) = H(0,0)f_{p+1}(0) = 0.
$$

Since by hypothesis $H(t,t) \neq 0$, we conclude that the derivative of $f_{p+1}(t)$ exists and is continuous and that $f_{p+1}(1) = f_{p+1}(0) = 0$. Hence it is necessary that the linear functional of continuity order $p$ reduce, by an integration by parts, to a linear functional of continuity order $p-1$. The lemma applied again to this new functional reduces it to a linear functional of continuity order $p-2$. Applying the lemma successively in such a manner, we finally get, as a necessary condition for invariance, the result that the original linear functional of continuity order $p$ has to reduce at least to a linear functional of continuity order one. This result coupled with the existence theorems cited in the beginning of § 5, establishes our theorem.

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# ON THE DISTRIBUTION OF QUADRATIC AND HIGHER RESIDUES*

BY H. S. VANDIVER

1. *Introduction.* In the present paper theorems will be obtained regarding the distribution of quadratic and higher residues. Special cases of the theorems yield results concerning the class number of quadratic forms of determinant $(-d)$ where $d \equiv 3 \pmod 4$.

2. *Conjugate Sets of Residues.* In a previous article† the writer considered the notion of *conjugate set* in a *finite algebra.* Applied to the finite algebra represented by the residue classes of a rational integer as modulus, we may define a *conjugate set of residues* of the modulus $m$ to be a set

$$(1) \qquad a_1, a_2, \cdots, a_i$$

* Presented to the Society, October 28, 1916.

† ANNALS OF MATHEMATICS, (2), vol. 18 (1917), p. 106.