COOLIDGE ON COMPLEX GEOMETRY

The Geometry of the Complex Domain. By Julian Lowell Coolidge. New York, Oxford University Press, American Branch, 1924. 242 pp.

The various geometries of a complex space are, with certain definite exceptions, essentially the same as the corresponding geometries of a real space, provided that one agrees, tacitly or wittingly, to admit to consideration only those configurations in complex space which are the direct generalizations of configurations in real space in that they arise from these by the replacement of real by complex numbers. This agreement restricts, for example, the discussion of manifolds of complex points to those manifolds depending on a fixed number of complex parameters. But a system of points depending on n complex parameters is but a very special case of a system depending on the equivalent number, $2n$, of real parameters, and its properties are in no way indicative of those of the general $2n$-parameter system. In other words, when the scope of complex geometry is widened to cover all possibilities, it takes on an entirely different aspect from that of real geometry.

Though there are recent books on complex geometry which pursue the subject along the narrow path outlined by the geometry of reals, the one now under review is the first to follow consistently the wider and, for the progress of mathematics, the more important, point of view. It aims to bring the reader abreast of the times in the advances, from this point of view, of the last half-century. The pioneers in these researches have been Segre and Study, and to them the book is informally dedicated. Among their disciples are to be mentioned Autonne, Benedetti, Coolidge, Fubini, Loewy, Sforza, and J. W. Young.

Closely connected with, if not strictly speaking a part of, complex geometry is the classical problem of representing complex points by real elements. An historical and critical account of the many attempts, ancient and modern, successful and unsuccessful, to solve this problem forms the second integral part of the book. The writers here are legion; we content ourselves with the names of the more important: Wallis, Wessel, Argand, Gauss, Poncelet, von Staudt, Laguerre, Marie, Klein, Segre, and Study.

The material has been well digested and excellently ordered. The complex line (Chapters I, II), the complex plane (Chapters III–VI), and complex three-dimensional space (Chapter VII) are taken up in turn. In each case, the real representations of complex points are first considered and then later put to work in throwing light on the complex geometry itself. The author writes in his characteristic, red-blooded style; his historical accounts and criticisms are, in particular, entertaining and stimulating. He has at various stages, notably in connection with the differential geometry of manifolds of points, contributed original results.

The developments of the wider aspects of complex geometry pertain largely to manifolds of points depending on a fixed number of real parameters. The study of algebraic manifolds and their projective properties has received the lion's share of attention and forms a dominant feature of the present book. The general analytic manifolds and their differential properties have been discussed only recently, by the author himself.

If $x_1, \cdots, x_n, x_{n+1}$ are complex variables and $\bar{x}_1, \cdots, \bar{x}_n, \bar{x}_{n+1}$ are their conjugates, a polynomial which is homogeneous and of degree m in each set of variables and has the property that it is a constant multiple of the conjugate-complex polynomial, is called a hyperalgebraic form. Using symbolic notation, we can write: $(ax)^m(\bar{a}\bar{x})^m$, where $(ax) = a_1x_1 + \cdots + a_nx_n + a_{n+1}x_{n+1}$. When the order, $2m$, of the form is equal to two, it reduces to the better known Hermitian form, $(ax)(\bar{a}\bar{x})$.

If we think of $x_1, \cdots, x_n, x_{n+1}$ as homogeneous coordinates of a point in a complex space of n dimensions, and set our hyperalgebraic form equal to zero, the locus of the equation obtained, if it has any locus other than singular points, is an algebraic manifold of the type just mentioned. The number of real parameters on which it depends is $2n - 1$, one less than the total number of real degrees of freedom of a point in the space in question. Manifolds coming under this head are the predominant algebraic manifolds in their respective spaces; among them, the simplest and most interesting are those represented by Hermitian forms.

With this orientation concerning the general scope of the book, we can proceed with greater appreciation to a detailed discussion of its contents.

The first chapter deals with the attempts to represent by real elements the complex points of a line. Outstanding is, of course, the classical representation of analysis, first discovered by Wessel and later rediscovered by Gauss and Argand. It is interesting to note that it was Gauss who first introduced the representation by means of points; Wessel and Argand contented themselves with vectors.

The projective geometry of the complex line is the image of the real geometry of inversion in the Gauss plane, and we can best outline the content of the second chapter from this point of view. To a circle or a straight line in the Gauss plane corresponds on the complex line a system of points depending on one real parameter, known as a chain. A chain is characterized, analytically, by the fact that it can be represented by a Hermitian form, and, geometrically, by the properties that the cross ratios of any four of its points are real and that it contains the point making with any three of its points any given real cross ratio. The direct circular transformations of the Gauss plane correspond to the collineations of the line, whereas the indirect circular transformations, when interpreted on the line, are called anti-collineations. Collineations preserve a given cross ratio, anti-collineations change it to the conjugate-complex value; both carry a chain into a chain, and, taken together, they are the only continuous one-to-one transformations of the complex line which preserve harmonic separation.

Among the many further developments of this chapter we mention only the study of algebraic threads. A thread is a system of points depending on one real parameter. An algebraic thread can be represented by a hyperalgebraic form. It corresponds in the Gauss plane to an algebraic curve, of order equal to that of the form. The same order is given to the thread and a geometrical significance is found for it. Class is also defined for the thread, by the application of a transformation on the line corresponding to the polar reciprocation in the circle $x^2+y^2+1=0$ in the plane.

In the projective geometry of the complex plane (Chapter V), we have, besides the chain of points, its dual, the chain of lines; the two are projectively equivalent, respectively, to the range of real points and the pencil of real lines. The projective equivalents of the real points (lines) of the plane play also fundamental roles; these are manifolds of points (lines) depending on two real parameters, known as chain congruences of points (lines). As in the previous case, we have here both collineations and anti-collineations.

Just as ordinary polar systems lead to conics, the anti-polarities yield the so-called hyperconics. These are represented by Hermitian forms. They consist, then, not of ∞^2 but of ∞^3 points. On the other hand, there are no curves which are entirely contained in one of them. A straight line meets one either in a chain of points, a single point, or nowhere; the case of intersection in a single point occurs only when the line contains its pole in the given anti-polarity, and so such a line is called a tangent. Discussion of further developments, of the theory of linear systems of hyperconics, and of the work on hypercurves, three-parameter systems of points represented by hyperalgebraic forms, we must forego.

The geometry of the subgroup of the group of collineations which possesses the Hermitian form $(x\bar{x})$ as a relative invariant is known as a Hermitian metric of elliptic type. This metric and the parabolic metric resulting from it in the usual way are developed in detail. In the elliptic case, to which we confine ourselves, the governing group depends on three complex and three real parameters. Distance and angle are always real. A geodesic thread turns out to be a particular type of chain, characterized by the fact that its points are, in pairs, conjugate with respect to the fundamental form. The Hermitian trigonometry of a triangle is found to be identical with that of ordinary elliptic trigonometry only in case the altitudes are concurrent. In the discussion of the metric properties of hyperconics, it is discovered that the general hyperconic has a whole chain of foci and a chain of corresponding directrices, and that the ratio of the sines of the distances of a point on the hyperconic to a focus and to the corresponding directrix is not merely constant, but the same for all foci!

A representation of the complex points of the plane which is peculiarly applicable to the projective and metric theories just reviewed, is that of Segre. Thereby, each point is represented by a real point of a four-dimensional variety of the sixth order in a space of eight dimensions. Through each point of this S_4^6 pass two conjugate imaginary planes of the S_4^6. The collineations and anti-collineations of the plane are represented in S_8 by real collineations, which, in the one case, permute the planes of each of these systems and, in the other, interchange the two systems. A straight line is represented by a quadric surface, a chain by a conic, a chain congruence by a surface of Veronese (Cayley), and a hyperconic by the intersection of the S_4^6 by a hyperplane. The Hermitian metrics appear on S_4^6 as classical non-euclidean metrics!

Chapters III and IV are devoted to a review of the various methods of representing the complex points on a curve and in a plane, respectively. Aside from the representation of Segre, just described, and that of Klein and Study, to which, for lack of space, we can give but passing mention, the most important of these methods are those of von Staudt, Marie, and Laguerre. According to von Staudt, an imaginary point is represented by the elliptic involution on the real line through the imaginary point which has as its double points the imaginary point and its conjugate, or, more accurately, by this involution provided with a sense. The Laguerre representation consists of the two real points, taken in a specified order, in which the isotropic lines through the given imaginary point intersect those through the conjugate-imaginary point. The Marie representation is also an ordered pair of points, lying on the real line through the given imaginary point. These three representations of an imaginary point, though apparently essentially different, are closely related. The points of the Laguerre representation, when rotated through a right angle about the mid-point of their segment, become those of the Marie representation; these are, in turn, a pair of points in the von Staudt involution, in fact, that pair whose distance apart is least.

The Laguerre and Marie representations of a curve have been exhaustively treated by Study. They both consist, in general, of real transformations of the plane, inversely conformal transformations in the one case, and special directly equi-areal transformations in the other. These results are developed in Chapter VI, on the differential geometry of the plane. The remainder of the chapter consists of the author's recently published work on the differential geometry of two- and three-parameter manifolds of points.

The geometry of three-dimensional complex space is considered in Chapter VII. The projective theory follows, generally speaking, the lines of that in the plane. The representation of Marie still consists of an ordered pair of points; when applied to a curve, it yields two (translation) surfaces corresponding by an elliptic parallel map which is equi-areal and of invariant -1.* The Laguerre representation is not so simple, consisting now of an oriented circle; the results thus far obtained concerning it are interesting and there is still room for further labors.

An interesting feature of this chapter is the geometry of the minimal plane (Beck, C. L. E. Moore). The chapter closes with some previously unpublished work of the author dealing with analytic manifolds depending on n real parameters, $n = 2, 3, 4, 5$, and establishing in particular the conditions that such a manifold contain one or more curves or surfaces.

A book on complex geometry can hardly be considered complete unless it establishes a logical foundation for the subject. The simplest foundation is an analytical foundation, based on the complex number system, and such a foundation is inherent in the opening chapters of the book. In the last chapter, the author carries through a more adventurous method, by means of postulates. A system of postulates for the real projective domain, that of his book on Non-Euclidean Geometry, is first developed and projective coordinates for real elements introduced. Imaginary elements are then defined by means of the elliptic involutions of von Staudt and finally acquire coordinates through the medium of von Staudt throws.

The reviewer has but one serious quarrel with the author. To be sure, he would have been happier if the author had followed Study in choosing the word membrane instead of the over-worked word congruence to designate a system of points depending on two real parameters. But this is a matter of minor importance. What caused much greater mental anguish was the author's indiscriminate use of the words complex and imaginary. He seems to have made no conscious

* Cf. Graustein, *Real representations of complex curves,* ANNALS OF MATHEMATICS, (2), vol. 26 (1924), p. 143. The previous description by the reviewer, reproduced by Coolidge, has the disadvantage that it employs imaginary curves (the generators) on the representing surfaces.

attempt to distinguish between the two and repeatedly uses complex as the opposite of real. In a book on complex geometry this is especially to be regretted.

The book is marred by many, very many, typographical errors. Those in the text itself are not seriously disturbing; in fact, some of them, as, for example, "any value linearly descendent on two given values" on p. 108, and "two intersectional minimal lines" on p. 190, are decidedly the opposite. But those in the formulas, though they may be only irritating to the mature reader, are likely to prove a stumbling block for the budding mathematician.

Aside from these matters, the book is full of solid, stimulating mathematics. Moreover, it is particularly welcome in that it brings up to date a field of geometry which is comparatively new and by no means exhausted. May it prove the inspiration and basis of departure for fresh endeavors.

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GAUSS AND NON-EUCLIDEAN GEOMETRY

A question of historical accuracy is raised by Professor Emch in his review of my *Projective Geometry** where he says:

"It is proper to point out an error which is common in histories of mathematics and which is contained in the following statement on page 420: 'Little progress was made until about a hundred years later when Gauss (1777–1855), his friends and pupils became deeply interested in the subject.' (1) Now the fact is that Gauss's deeper interest in the subject was subsequently aroused by the brilliant discoveries of Lobatchevsky and Bolyai. (2) As a matter of fact, Gauss, in the beginning, hoped to be able to prove what is known as the euclidean parallel axiom and (3) assumed a rather skeptical attitude towards the new discoveries. (4) Subsequent deeper meditations, however, led Gauss to his own establishment or verification and acceptance of the new theory."†

An examination of the sources on which my statement is based will, I believe, substantiate the sentence quoted as well as the context from which it is taken. Among these sources are the letters of the

* This BULLETIN, vol. 30 (1924), p. 81.

† The numbering is mine.