JULIA ON ESSENTIAL SINGULARITIES

Leçons sur les Fonctions Uniformes à Point Singulier Essentiel Isolé. By Gaston Julia. Redigées par P. Flamant. (Borel Monograph.) Paris, Gauthier-Villars, 1924. vii + 149 pp.

This monograph deals with Picard's famous theorem of 1879 to the effect that an analytic function, in the neighborhood of an isolated essential singularity, assumes every value, with two possible exceptions, an infinite number of times. It presents this theorem, and perfections of it due to Landau, Schottky, Caratheodory, Lindelöf, Iversen, and to Julia himself.

A first chapter develops as much of the theory of modular functions as is needed for the proof of Picard's theorem. The theorem that any simply connected region can be mapped conformally upon a circle is here used, and familiarity with it by the reader is assumed. Of course, an elementary proof of Picard's theorem, by the methods of Borel, Landau and Schottky, would not require as much preliminary work, but when one considers the intuitive qualities of Picard's own proof, one can understand Julia's preference. Besides, as Caratheodory has shown, (this is dealt with in the second chapter), the modular function does not play an artificial rôle, for it leads to the determination of an important least upper bound, in the expression for which the modular function actually appears.

The second chapter opens with the proof of Picard's theorem for the special case of an integral function. Then comes the following generalization, due to Landau:

There exists a function $\varphi(a_0, a_1) > 0$, *such that, if the series*

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots, \tag{1}$$

with $a_1 \neq 0$, *converges for* $|x| < \varphi(a_0, a_1)$, *the series assumes at least one of the values* 0 *or* 1 *for* $|x| < \varphi(a_0, a_1)$.

Caratheodory's determination of the best $\varphi(a_0, a_1)$ is then given. As stated above, the modular function appears in this expression. The next theorem is that of Schottky:

There exists a function $M(a_0, \theta)$, *such that, if* (1) *converges for* $|x| \leqq R$ *and does not assume either of the values* 0 *or* 1 *for* $|x| \leqq R$, *we have, for any positive* θ *less than unity, and for* $|x| < \theta R$, *the inequality* $|f(x)| < M(a_0, \theta)$.

The chapter closes with the proof of Picard's theorem for any isolated essential singularity.

The third chapter, one of the most interesting in the book, gives an account of Montel's *normal families* of functions, which Julia used in his own researches. A set of functions, meromorphic in an area, is said to be *normal* in the area if from every sequence of functions of the family a sub-sequence can be extracted which converges uniformly in every area interior to the given one. Here the notion of uniform convergence has to be loosened somewhat, because of the poles. Also, the infinite constant is an allowable limit for the sub-sequence.

Montel's chief theorem states that *a sufficient condition for a family to be normal in an area is that three distinct values, a, b and c exist, none of which is assumed in the area by any function of the family.*

Montel's work on normal families has been singularly fruitful in many connections. One might mention conformal mapping, iteration and Picard's theorem. Still it seems to the present reviewer that Montel's principal theorem is little more than a combined statement of Schottky's theorem and of well known results on equally continuous families of functions.

An interesting detail of the third chapter is the crediting of M.B. Porter with the independent discovery of the theorem on sequences of analytic functions usually known as Vitali's theorem.

Chapter IV deals with researches of the Scandinavian school of analysts on sectors with vertex at an essential singularity. We pass over these results.

We come now to Julia's own work, which occupies the last three chapters. Julia's first problem amounts to this: *If $f(z)$ is a meromorphic function, and if $|\sigma| > 1$, what can be said as to the values assumed by the sequence of functions $f(\sigma^n z)$, $n = 1, 2, \cdots$, in any given area?* Those familiar with the recent work on the iteration of rational functions will see a relation between this problem and the theory of Poincaré's functions with rational multiplication theorems, which functions were introduced into iteration theory by S. Lattès and by the present reviewer. Julia's work deals principally with the set of points at which the family $f(\sigma^n z)$ is not normal. The results are numerous and interesting. In the final chapter, a set of multipliers not of the form σ^n is used. As a capital yield of Julia's methods, we cite the following extension of Picard's theorem for integral functions: *Given an integral function $f(z)$, and any point of the plane, there exists a ray joining the point to infinity such that, in any sector containing the ray in its interior, $f(z)$ assumes every finite value, with one possible exception, an infinite number of times.*

Julia's monograph is probably one of the finest in the Borel collection. The student who reads it together with Landau's *Neuere Ergebnisse der Funktionentheorie* will acquire dominion over a region of analysis whose exploitation is not yet complete.

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